Bayes-Nash Equilibria in Generalized Second Price Auctions with Allocative Externalities

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Abstract

In this paper, we investigate an incomplete information model of generalized second price auctions with allocative externalities originating from the heterogeneous match rates of bidders. A novel feature of our model is that it generates endogenous click-through rates (CTRs). In this setting, we establish existence of symmetric efficient equilibria for common classes of primitives. This contrasts with the findings of Gomes and Sweeney (2014), who study a similar model but with fixed CTRs. Moreover, non-existence results require strong assumptions on the primitives of the model. We conclude that existence of equilibria in GSP with incomplete information is quite general.

Keywords: Generalized second-price auction, Bayes–Nash equilibria, Position auctions, Sponsored search, Click–through rates.

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1 Introduction

Since the introduction of the internet, online advertising has become one of the major revenue sources for the IT industry (in particular, search engines) and growth has been explosive. For example, in 2005 Google's annual advertising revenue totalled about \$6.1 billion; by 2014 this figure had increased to \$59.6 billion, in both cases accounting for about 90% of total revenue. The enormous income was generated mostly by auctioning off millions of keywords searched by users. A popular auction format in practice is the so-called "Generalized Second Price" (GSP)[1] auction, which has recently received a great deal of attention from computer scientists and economists.

GSP is a special format of multiple-object auctions, in which a bidder wins at most one single object. In terms of a keyword auction, the objects are ranked sponsored links on the particular search result web page. Every bidder bidding for the keyword submits one single bid in order to have his link listed on that page. In the basic version of GSP, the bidder with the highest bid wins the highest position on the web page and pays the second highest bid whenever his link is clicked.[2] The bidder with the second highest bid has his link listed in the second position and pays the third highest bid when the link is clicked, and so on.

Most studies of GSP are based on the assumption of fixed click-through rates (CTRs) for positions, i.e., for a certain keyword and for any permutation of advertisers' links, each position receives a fixed and different proportion of clicks from a unit measure of users who are searching for the keyword. It is well-known that, unlike in single-object second price auctions, in GSP auctions truth-telling is no longer the dominant strategy and bidders have incentives to shade their bids.[3] Even more strikingly, regardless of the distribution of types, there is a wide range of model parameters with which bid-shading may lead to non-existence of symmetric and efficient equilibrium (simply equilibrium or efficient equilibrium from now on) in a Bayesian game setting. For example, Figure 1 from Gomes and Sweeney (2014) documents that in a 3-bidder-2-position GSP auction, the candidate equilibrium bidding strategy is not monotone and efficient equilibrium does not exist, if the CTRs are fixed and the second position's CTR (c_2) is greater than $\frac{3}{4}$ of the CTR of the first position (c_1) and advertisers' values (v) are independently and uniformly distributed on the unit interval. The intuition for such results is that if the CTRs are fixed and the second position's CTR is sufficiently close to the first position's CTR, then the two positions have virtually the same value for bidders. By shading bids intensively so as to win the second position, bidders receive almost the same amount of clicks but pay a much smaller amount for each click. Consequently equilibrium may not hold due to the intensive bid-shading behavior of bidders.

However, for sponsored link auctions, the assumption of exogenous CTRs is too strong given the nature of keyword searches, i.e., for a certain keyword, the products being advertised are typically close

[1]The terminology was proposed by Edelman et al. (2007).
[2]For weighted GSP, see section 6 for a discussion of quality score.
[3]See, e.g., Edelman et al. (2007) for an illustration using a small example.

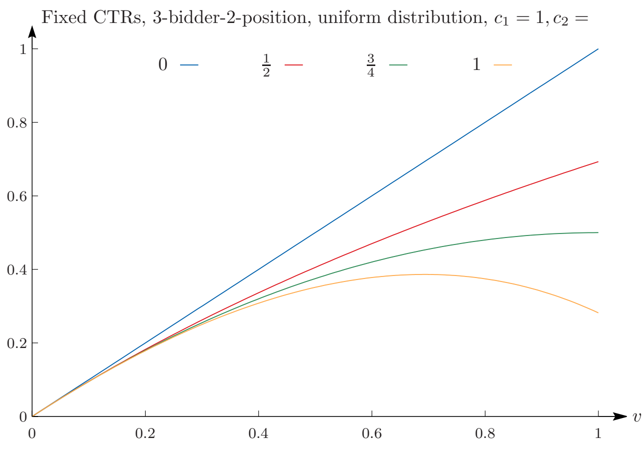

Figure 1: Non-monotone candidate equilibrium bidding strategy in the fixed-CTR model of Gomes and Sweeney (2014)

substitutes since the meaning of keyword is *precision*.[4] A keyword typically implies a specific interest and a user typically has a unique demand for a certain product. On the other hand, the probability of different advertisers meeting a particular user's need, or *match rates*, is variable. That is, for a certain advertisement, a certain proportion of users will find that the product advertised satisfies their needs, and consequently will not continue to spend time and energy searching for the contents of other links. Therefore, the CTR of a particular position naturally depends on the match rates of advertisers on the other positions. Put another way, different advertisers exert different degrees of *allocative externalities*[5] to each other and the CTRs are endogenous.

The externalities generated by match rates are key to forming bidding strategies and consequently the existence of equilibrium. In the fixed-CTR model, the expected CTRs of the positions are unconditional. In contrast, with allocative externalities, bidders have to perceive the expected CTRs of the positions as conditional. More specifically, in an environment with allocative externalities and assuming a top-down browsing strategy by users[6], higher match rates from advertisers in top positions lead to lower CTRs in lower positions, making lower positions less valuable. When advertisers bid, they must take externalities into account. In particular, as can be seen in Figure 1, non-monotonicity in the fixed-CTR model is due to the bid-shading behavior of high-type bidders when the CTR of the second position is high. When match rates are taken into account, high-type bidders have less incentive to shade their bids in an efficient equilibrium.[7] Intuitively, if a high-type bidder is allocated to the second position, the bidder on the first position will have an at least equally high match rate resulting in a low CTR for the second

[4]when a user searches for a desired product, she would typically and can effortlessly use a combination of words to search for the most relevant products. The combination is interpreted as a keyword.

[5]See, e.g, Jehiel and Moldovanu (2006).

[6]A justification of a top-down strategy by consumers is provided in section 2.

[7]See section 2 for a definition of efficient equilibrium.

position. Indeed, as shown below, for bidders with match rates greater than one-half the bid function is always monotone. As for low-type bidders, they would have been striving to get on the list by bidding aggressively. Therefore, we may expect that in general the candidate equilibrium bidding strategy is monotone and equilibrium exists.

Due to the sealed-bid policies of search engines and the entry and exit of advertisers, the match rates of advertisers are typically private information. It is well-known that the existence of Bayes-Nash equilibria in auctions critically depends on the monotonicity of the candidate bid functions, which in turn depends on the primitives of the model, i.e., the distribution of types and model parameters etc. To ease the exposition, we focus our analysis on the N-bidder-2-position case. We show in section 4 that equilibrium exists for familiar distributions. For example, equilibrium exists for "normal like" distributions. In addition, if the density function is non-decreasing, then equilibrium also exists. In order to illustrate further the generality of the existence of equilibrium, we consider some particular families of distributions. For distributions with decreasing density functions, we find that equilibrium exists if they belong to the power family.[8] It can also been shown analytically that equilibrium holds for left-skewed and inverse U-shaped Beta distributions, i.e., with parameters of $a \geq b$. Using numerical methods, we find that equilibrium also exists for right-skewed Beta distributions as long as the distance between b and a is not too large.

It seems that the non-existence of efficient equilibria is an inherent property of GSP with continuous action space and incomplete information. In section 5 we give examples where the monotonicity of the bidding function breaks down and efficient equilibria fail to exist. However, we believe that cases of non-existent results in our model have quite different features and are practically marginal compared to the fixed-CTR model in Gomes and Sweeney (2014). First, as illustrated in section 5, they happen only for distributions with extreme parameters, e.g., Beta distributions with parameters $a = 2$ while $b > 70$ approximately[9] or uniform distributions with partial support on $[0, b]$ and $b < 0.15$ approximately, which all require strong assumptions on the primitives. In contrast, non-existence is ubiquitous in the fixed-CTR model, i.e., across all distributions of types.[10] Second, in our model, non-existence occurs when the mass of the distribution is concentrated on close-to-zero values, while in the fixed-CTR model it occurs with higher intrinsic values of positions (e.g., $c_2 > \frac{3}{4}c_1$). Recalling that non-existence is due to the intensive bid-shading behavior of bidders, the results in the fixed-CTR model appear to be disruptive (for search engines) in the sense that more valuable objects result in lower revenues from the auction. Moreover, if, e.g., the distribution of c_2 is uniform in large-scale auctions, then up to $\frac{1}{3}$ of the revenue may be lost compared to an ideal situation where equilibrium always exists. Most importantly, the intuition for the non-existence results in our model is that the externality effects almost disappear when the auction is full of low-type bidders. However, an empirical study by Gomes et al. (2009), "suggests that externality

[8] Thus, for the entire power family, equilibrium always exists.

[9] See Appendix B for the definition of the Beta distribution.

[10] See Proposition 2 in Gomes and Sweeney (2014).

effects are indeed economically and statistically significant". In summary, cases of non-existence results are likely to be of marginal importance in practice once allocative externalities are appreciated by bidders.

The rest of the paper is organized as follows. First we review briefly the related literature and then in section 2 the model is set up. Section 3 characterizes the efficient equilibrium when it exists, and the necessary and sufficient conditions are given for existence of the equilibrium. The main results in section 4 shows that existence is indeed quite general in terms of the primitives when externalities are taken into account. Section 5 gives some particular examples of non-existence, which show that non-existence requires very special and strong assumptions on the primitives and hence is practically marginal. In section 6 we discuss various extensions to the model, e.g., incorporating consumer search costs, allowing for heterogeneous profits per match and quality score adjustments of bids and payments by search engines. In addition, we show the revenue equivalence to other standard auctions, e.g., the Generalized English auction and the Generalized First Price auction. The optimal reserve price is identified, which is determined by the hazard rate function. We also notice that if there are no symmetric efficient equilibria then there are no symmetric inefficient equilibria either. Section 7 concludes.

Related literature

Assuming fixed CTRs for each position and the homogeneous valuation of bidders across the positions, Edelman et al. (2007) characterize *Locally Envy Free* equilibria for GSP in a complete information setting and unique perfect Bayesian equilibrium in a corresponding Generalized English auction. Using a different approach, Varian (2007) derives similar results. Börgers et al. (2013) assume fixed advertiser-specific CTRs and relax bidders' preferences to allow for position-specific valuations. They prove the existence of symmetric Nash-equilibria and conduct an empirical analysis on bidders' willingness-to-pay. Still under the assumption of fixed CTRs but in an incomplete information setting, Gomes and Sweeney (2014) derive the symmetric efficient Bayes-Nash equilibrium bidding strategy and the optimal reserve price when equilibrium exists. Their most striking result is that in an incomplete information setting a symmetric model with a wide range of parameters may have no symmetric equilibria for any distributions. Hence, their work is closest to ours and our work complements their study of GSP in the direction of incomplete information setting.

The assumption of fixed CTRs may apply when the match rates of advertisers are all the same. But then the CTRs decreases in a constant proportion over positions, which is not realistic. The fixed-CTRs assumption may also apply when consumer behavior is abstracted in such a way that they actively search for products in order to find the product best matching their interest but at the same time act like robots, which is not reasonable. Otherwise, endogeneity arises naturally, and is often generated by so-called *continuation probability* in the literature. Continuation probability, as an attribute of a certain advertisement, is the probability that, after clicking on the advertisement, a representative consumer will continue to browse or click on the next advertisement. In principle, a less-than-one continuation probability may be due to the fact that some consumers either find their desired products on the link and stop searching, or expect much lower match rates for the remaining links and discontinue the searches. Hence, $(1 - match\ rate)$ in our model can be thought as a component of continuation probability. Without considering a particular auction format, Aggarwal et al. (2008), Ghosh and Mahdian (2008) and Kempe and Mahdian (2008) all study winner determination with different extensions in cases where CTRs are endogenous due to externalities generated by continuation probability and the objective of the seller is to maximize the aggregate advertiser surplus. Using a similar setting, Giotis and Karlin (2008) characterize the complete information Nash equilibria in GSP.

Jehiel and Moldovanu (2006) study effects of allocative and informational externalities in general and find that allocative externalities may lead to multiple equilibria and welfare maximization is generically impossible to be implemented via ex-post equilibrium when informational externalities are present. Athey and Ellison (2011) consider a model of position auctions similar to ours, except that consumer search costs are assumed to be positive. With heterogeneous match rates and positive consumer search costs, both kinds of externality are present in the auctions. However, the specific auction format in their paper is a version of Generalized English auction and their main focus is on welfare, e.g., the sorted advertisement list in the efficient equilibrium improves consumer welfare and typically social welfare as well. Still assuming positive consumer search costs but allowing advertisers to be heterogeneous in both their "relevance" to consumers and valuation per click, Gomes (2014) derives the optimal auction design from a two-sided market perspective. The "relevance" of an advertisement is quite similar to the match rate in our work, but only one position is studied in Gomes' paper and thus there is no externalities taken into account in the auctions. Based on a special environment with complete information of match rates, Chen and He (2011) analyze the welfare and revenue effects of advertisers' pricing and bidding decisions in GSP. Caragiannis et al. (2015) delimit the price of anarchy (i.e., efficiency loss) of GSP in both incomplete and complete information settings.

2 The model

A search engine is auctioning off a list of K advertisement positions for a certain keyword. There are N $(> K)$ bidders and each of them submits a single bid in order to get on the list. Each bidder independently draws a match rate from a common atomless distribution $F[0, 1]$. A bidder with a match rate of m has a probability m of meeting a consumer's need when the consumer clicks on his link. Bidders' profits from each match are assumed to be the same and normalized to 1. As suggested in the empirical study of sponsored link auctions by Celis et al. (2014), "bidder valuations are private, driven by idiosyncratic match quality". Hence, the assumption of homogeneous profits may be a good approximation. [11] Comparing to Gomes and Sweeney (2014), we may say that here the payoffs from one click are also independently and identically distributed according to $F[0,1]$.

The auction format is GSP, and the selection rule is such that the highest K bidders are listed on the page and bidders are ranked by their bids. The bidder on position k pays β_{k+1}, the bid of the bidder on position $k+1$, for every click he receives. Hence, the net payoff from one click is $m-\beta_{k+1}$ if a type m bidder is listed on position k.

A unit measure of consumers have unit demand and search with top-down strategy by which they stop searching if they are matched with a certain link's product(s) and otherwise always continue to click on the next link. Thus the first link always receives a measure of 1 of clicks. In expectation, the second link receives $1-m_1$ clicks if the first link has a match rate m_1, and the third link receives $(1-m_1)(1-m_2)$ clicks if the second link's match rate is m_2, and so on.

Since the model is symmetric, a natural equilibrium candidate is the pure strategy *symmetric efficient equilibrium* in which each bidder uses the same bid function and is ranked by his match rate. The top-down search strategy of consumers is rationalized in efficient equilibrium. That is, given the efficient allocation of positions to advertisers in which bidders with higher match rates are listed in higher positions, the best response of consumers is to search top-down as long as a small search cost is considered.

3 The Efficient Equilibria

In the efficient equilibria, bidders are assigned probabilities of being listed on positions according to their match rates. Thus, a bidder with match rate m is assigned to position k with the probability

$$\pi_k(m) = \binom{N-1}{k-1}(1-F(m))^{k-1}F(m)^{N-k}. \tag{1}$$

Because all consumers use a top-down strategy, the expected number of clicks on position 1 is 1 for any m. The expected number of clicks received by a bidder with match rate m, if he is on position $k \geq 2$, is

$$\begin{aligned} c_k(m) &= E\left[\left(1-F^{1:N}\right)\cdots\left(1-F^{k-1:N}\right)\middle|F^{k:N}=m\right] \\ &= \int_m^1\cdots\int_m^{t_{k-2}}(1-t_1)\cdots(1-t_{k-1})\frac{f^{k-1:k-1}(t_{k-1})}{1-F(m)}\cdots\frac{f^{1:k-1}(t_1)}{1-F(m)}\,\mathrm{d}t_{k-1}\cdots\,\mathrm{d}t_1 \end{aligned} \tag{2}$$

where $F^{k:N}$ is the k_{th} highest order statistics of N independent draws from distribution F, and the density is

$$f^{k:N}(t) = \frac{N!}{(N-k)!(k-1)!}F(t)^{N-k}(1-F(t))^{k-1}f(t).$$

[11] Athey and Ellison (2011) make the same assumption in their version of Generalized English auctions with consumer search. See section 6 for a brief discussion of allowing for heterogenous profits.

Having identified $\pi_k(m)$ and $c_k(m)$, we are in a position to give the necessary and sufficient condition for the existence of the efficient equilibrium.

Proposition 1. *With K positions and $N \geq K$ bidders with I.I.D match rates drawn from $F[0,1]$, a unique efficient equilibrium exists if and only if the GSP auction possesses a symmetric bidding strategy which is given in equation (3) and is strictly increasing in match rate.*

$$\beta(m) = \Phi(m) + \sum_{n=1}^{\infty} \int_0^m \Phi(t) Q_n(m,t)\, \mathrm{d}t, \tag{3}$$

where

$$\Phi(m) = \frac{\frac{1}{N-1}\sum_{k=1}^{K}\left[\frac{\mathrm{d}\pi_k(m)}{\mathrm{d}m}c_k(m) + \pi_k(m)\frac{\mathrm{d}c_k(m)}{\mathrm{d}m}\right]m}{\sum_{k=1}^{K} c_k(m)[1-F(m)]^{k-1}\binom{N-2}{k-1}F(m)^{N-k-1}f(m)},$$

$$Q_1(m,t) = \frac{\sum_{k=1}^{K}\left\{c_k(m)(k-1)f(m) - \frac{\mathrm{d}c_k(m)}{\mathrm{d}m}\left[1-F(m)\right]\right\}(1-F(m))^{k-2}\binom{N-2}{k-1}F(t)^{N-k-1}f(t)}{\sum_{k=1}^{K} c_k(m)\left[1-F(m)\right]^{k-1}\binom{N-2}{k-1}F(m)^{N-k-1}f(m)},$$

$$Q_n(m,t) = \int_0^m Q_1(m,\xi)Q_{n-1}(\xi,t)\,\mathrm{d}\xi, n \geq 2,$$

if

$$\Phi \in L^2[0,1] \text{ and } \int_0^1\int_0^1 |Q_1(m,t)|^2\,\mathrm{d}m\,\mathrm{d}t < \infty.$$

Proof. See appendix A. □

Example 1. *3-bidder-2-position, uniform distribution*

When there are 3 bidders and 2 positions and $F(m) = m$, differentiating equation (8) in appendix A and rearranging it gives a first order differential equation for the equilibrium bid function

$$\beta'(m) - \frac{3m^2 - 6m + 1}{(1-m)(1+m^2)}\beta(m) - \frac{-6m^3 + 11m^2 - 4m + 1}{(1-m)(1+m^2)} = 0.$$

Even in this simplest case, the first order differential equation cannot be solved in tractable closed-form. Based on numerical methods, figure 2 shows that the bidding strategy is strictly monotone. As verified in the next section, the auction does have an efficient equilibrium. □

4 Existence of equilibrium when $K = 2$

As example 1 shows, even with the simplest parameterization, the candidate bid function is complex enough. On the other hand, for the purpose of showing the generality of existence of equilibrium based on the primitives of the model, it is instructive to study the case with only two positions. The following proposition identifies some sufficient conditions on the primitives for the existence of efficient equilibrium when $K = 2$.

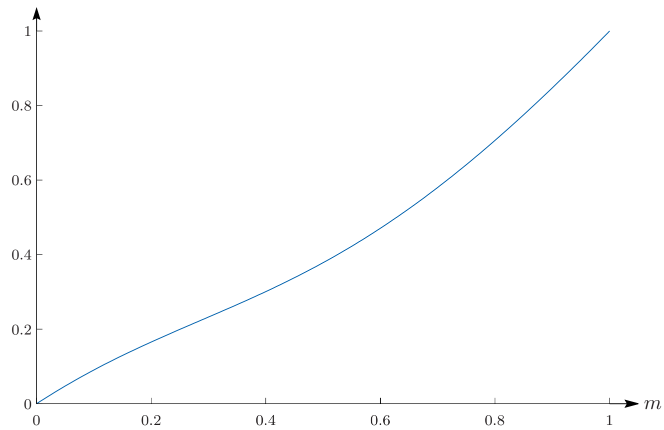

Figure 2: Equilibrium bid function in a 3-bidder-2-position GSP with uniform distribution

Proposition 2. *When $K = 2$, efficient equilibrium exists if the density function of the match rates satisfies $\Lambda(m) \equiv \int_m^1 (1-t)f(t)\,\mathrm{d}t - m(1-2m)f(m) \geq 0$ for any $m \in [0,1]$. In particular, $\Lambda(m) \geq 0$ as long as one of the following conditions holds:*

(i) *$f(m)$ is inverse-U shaped.*

(ii) *$f(m)$ is non-decreasing.*

(iii) *$f(m)$ is decreasing and belongs to the power family.*

(iv) *$f(m)$ is U shaped and belongs to Beta family.*

(v) *$f(m)$ is left-skewed and belongs to Beta family.*

Proof. See Appendix B. □

The second item in proposition 2 confirms that GSP does have an efficient equilibrium when the match rates are uniformly distributed as in example 1. As a more intuitive summary, Figure 3 illustrates that the existence is quite general for familiar distributions. For right-skewed distributions (the "missing" plot on the north-east corner in Figure 3), e.g., Beta distributions with parameters $a < b$, numerical approximation can be used to show that as long as the distance between b and a is not too large (e.g., $b < 70$ approximately when $a = 2$), equilibrium also exists.

The intuition for such general existence results is best understood by dividing bidders into 3 groups, i.e., with low, medium and high types. Intuitively, due to the presence of externalities to each other, high-type bidders have less incentive to shade bids because losing the first position induces a severe punishment on the number of clicks they will receive. On the other hand, very low-type bidders (in the neighborhood of zero) also have strong incentive to bid aggressively because their objectives are simply

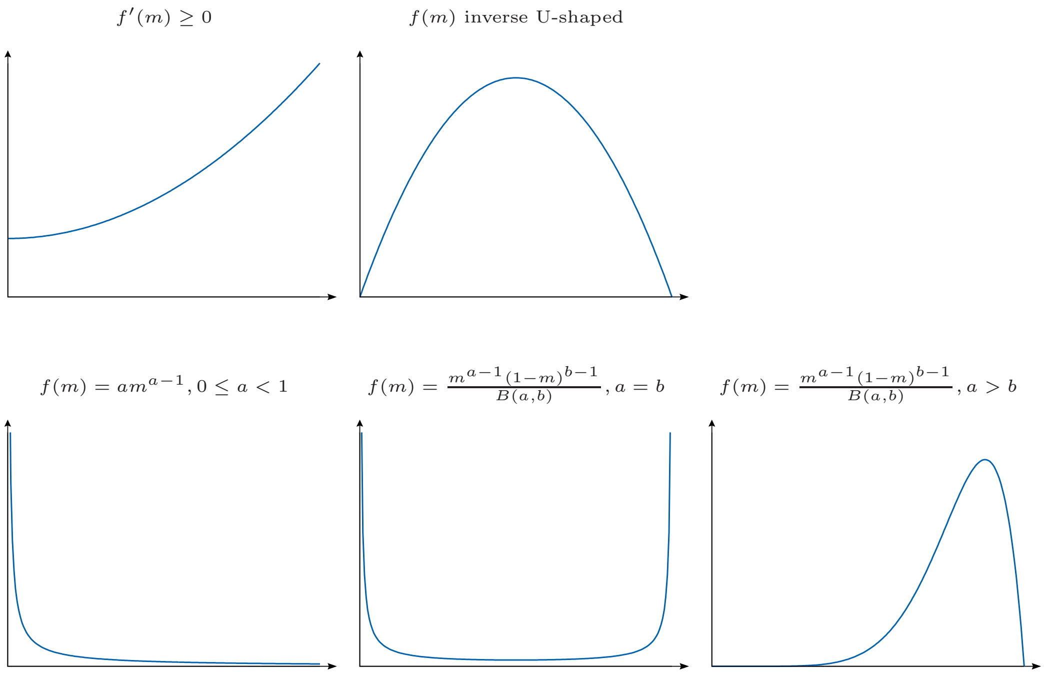

Figure 3: Familiar density functions leading to the existence of the efficient equilibrium, when $K = 2$

to maximize the probability of winning a position. Given the aggressive bidding behavior of both the high and low-type bidders, medium-type bidders are forced to bid aggressively as well, as long as the expected externalities from other bidders (especially the high-type bidders) are not immaterial. Hence, efficient equilibrium typically exists. On the contrary, in the fixed-CTR model, there are no externalities from other bidders. In particular, high-type bidders suffer only small and fixed punishments in the number of clicks from bid-shading when the gap between the CTRs of two positions is fixed at a low level. Therefore, it is the high-type bidders who have the incentives to shade bid intensively at the first place and consequently the non-existence results are ubiquitous in that model.

The next section provides examples where the expected externalities almost disappear and efficient equilibrium fails to exist for some distributions with extreme parameters when the number of bidders is small. Nevertheless, The following proposition shows that a large enough number of bidders always restores monotonicity and existence of efficient equilibrium.

Proposition 3. *If the number of bidders is large enough, for example, $N - 3 \geq \left\lceil \max_{m \in [0, \frac{1}{2}]} \frac{(1-2m)F(m)}{\int_m^1 (1-t)f(t)\,\mathrm{d}t} \right\rceil$ for all $m \in [0, \frac{1}{2}]$ when $K = 2$, then a symmetric efficient equilibrium always exists.*

Proof. See Appendix C. □

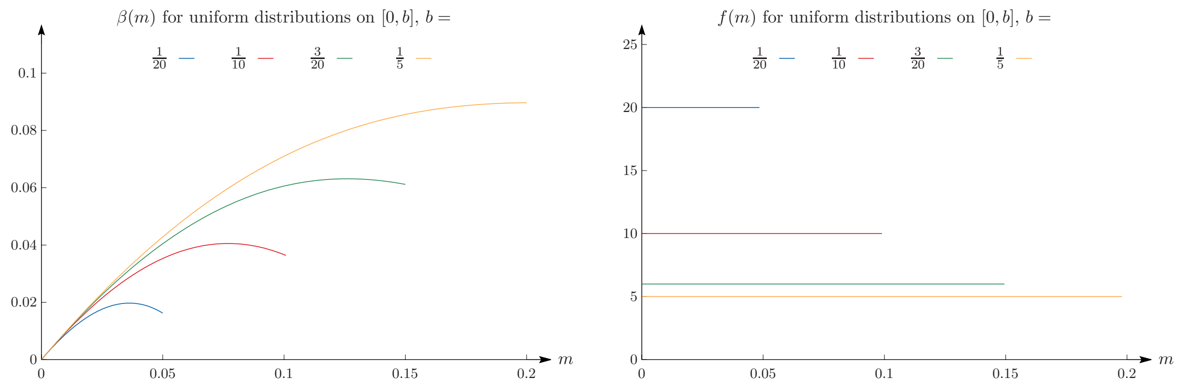

Figure 4: The candidate bid functions for uniform distributions on $[0, b]$, when $N = 3$ and $K = 2$.

5 Non-existence of efficient equilibria

The existence results in Proposition 2 are established for distributions with full support on $[0, 1]$. If instead, the match rates are distributed only on a partial support, e.g., $[0, b]$ where $0 < b < 1$, then monotonicity may fail. Indeed, as shown in appendix D, there is no efficient equilibrium if $F(m)$ is uniformly distributed on $[0, \frac{1}{10}]$ and there are only 3 bidders for 2 positions. Figure 4 shows that some of the bid functions are non-monotone for some small b.

With full support, if the density function is extraordinarily high for some small m, e.g., Beta distribution with parameter b significantly larger than a, then monotonicity may break down as well. Figure 5 gives a numerical approximation of this.[12]

Interestingly, the bid-shading behavior has different patterns in cases of full support and partial support. Non-monotonicity is due to the intensive bid-shading behavior of the bidders at the bottom of the distribution when it has a full support; on the contrary, when the distribution only has a partial support the bidders at the top of the distribution have incentive to shade bid at the outset such that the monotonicity breaks down. Intrinsically however, bidders with non-monotone bidding strategies are all of low, but not around zero, types. As explained in the previous section, the bidders with very low and very high match rates are always willing to bid aggressively.[13] When the probability mass on high match rates is small enough, the expected externalities from bidders with high match rates become virtually zero. Then, for bidders with low-but-not-around-zero match rates the situation becomes similar to the one in the fixed-CTR model. Winning the second position by intensive bid-shading may become profitable again and efficient equilibrium may not hold.

Recall from Proposition 2 that for power distributions with decreasing densities the bid function is always monotone. That indicates that non-monotonicity results not only require a concentration of prob-

[12]The derivatives of the bid functions are plotted here because non-monotonicity is almost indistinguishable from the bid function.

[13]A common feature of the equilibrium bid functions for the case of full support is that, as the one in Figure 2, they start with a concave curvature at the bottom of the distribution and becomes convex at the top.

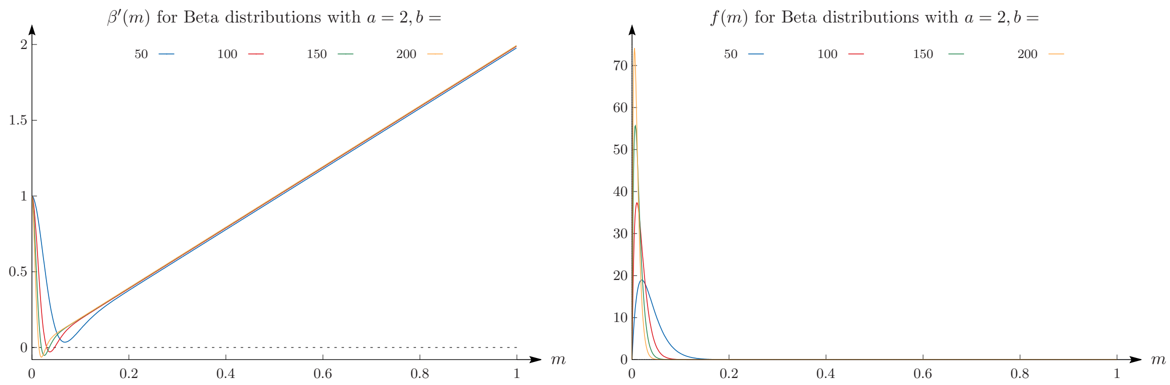

Figure 5: The derivatives of the candidate bid functions for Beta distributions with extreme parameters, when $N = 3$ and $K = 2$.

ability mass on low match rates, but also a special pattern of the density function. As proposition 2 shows, as long as $\Lambda(m) \equiv \int_m^1 (1-t)f(t)dt - m(1-2m)f(m) \geq 0$ for any $m \in [0, \frac{1}{2}]$, the candidate bid function is always monotone. Thus, non-monotonicity typically requires that the density function $f(m)$ peaks at some small m and at the same time for $\tilde{m} > m$, $f(\tilde{m})$ decreases at a sufficiently fast speed as $\tilde{m}$ increases, e.g, the Beta distribution cases in the right panel of Figure 5. In the extreme, the distribution may only have a partial support at low values, e.g., the cases in Figure 4. Because such scenarios require strong assumptions on the primitives, we may expect that they are unlikely to occur in practice, and the economic consequences will be marginal since the auctions are full of low-type bidders. This is in sharp contrast to the fix-CTRs model where in auctions with no equilibrium and consequently low revenue the objects have higher intrinsic values.

6 Discussion

To highlight the effects of allocative externalities on equilibrium, many aspects of GSP auctions are abstracted away from our model. In this section, we briefly discuss some potential extensions which may have effects on the generality of the existence of equilibria, e.g., incorporating consumer search costs, allowing for heterogeneous profits per match and introducing quality score adjustments which are popular in practice. In addition, revenue equivalence is obtained. The optimal reserve price is also indicated and its role in equilibrium existence is discussed by means of numerical methods . We also notice that in our model if there are no symmetric efficient equilibria then there are no symmetric inefficient equilibria either.

Consumer search costs

In our model, the allocative externalities suffered by a bidder are increasing in the match rates of the bidders on the higher positions. There may also be other types of externalities. For example, a poor quality or low match rate of an advertisement may cause consumers to cease further browsing, if consumer search is costly and consumers search optimally.[14] This kind of externality, which is decreasing in bidders' types, is called *informational externality*.[15] Notice that in our model the consumers have the lowest search cost, zero for everyone. Hence, for any non-degenerate distributions of search costs, not all of the unmatched consumers will be willing to continue to the next position, which will always increase the gap of the CTRs between two positions. This implies that lower positions always have less value, and the advertisers have even less incentive to shade bids. In fact, such informational externalities work exactly in the opposite direction of allocative externalities, i.e., the lower a match rate a bidder has, the larger the expected externalities he faces. This creates a problem similar to adverse-selection even for "medium" bidders who are again forced to bid more aggressively in order to avoid such externalities. Thus, the existence of equilibrium tends to be more robust, i.e., equilibrium will exist for an even wider range of distributions of match rates. For example, if consumers are sophisticated enough to recognize the match rate of each advertisement, it can be easily shown that efficient equilibrium exists for any distribution of match rates if consumer search costs are uniformly distributed in the 3-bidder-2-position case. The interim expected payment from the actual auction is then

$$\begin{aligned} E[P^{gsp}(m)] = 2\Big\{ &\int_0^m \int_0^{m_1} \beta(m_1) f(m_2) f(m_1) dm_2 dm_1 \\ &+ \int_m^1 \int_0^m (1-m_1) m_1 \beta(m_2) f(m_1) f(m_2) dm_2 dm_1 \Big\}. \end{aligned}$$

In a corresponding direct truthful mechanism, denoting the expected payment by $E[P^{DT}(\cdot)]$,

$$\begin{aligned} m \in \arg\max_{\hat{m}} 2\Big\{ &\int_0^{\hat{m}} \int_0^{m_1} m f(m_2) f(m_1) dm_2 dm_1 \\ &+ \int_{\hat{m}}^1 \int_0^{\hat{m}} (1-m_1) m_1 m f(m_2) f(m_1) dm_2 dm_1 - E[P^{DT}(\hat{m})] \Big\}. \end{aligned}$$

The same procedure as in Appendix A and B gives the equilibrium bid function which can be shown to be monotone for any distributions of match rates.

Heterogeneous profits per match

The bidders' profits per match are assumed to be homogeneous in our model, which is restrictive, but allows us to highlight the importance of allocative externalities induced by match rates. With heterogeneous profits per match, e.g., independently and identically distributed according to $V \sim G[0, \bar{v}]$, the

[14] In Athey and Ellison (2011) such search costs are incorporated and consumers are allowed to do Bayesian updating of their beliefs about the expected match rates of advertisers in the generalized English auction .

[15] See e.g., Jehiel and Moldovanu (2006).

social efficient allocation is typically not monotone in either m or v or even $\omega \equiv mv$. Using a similar setting, Aggarwal et al. (2008) and Kempe and Mahdian (2008) provide algorithms to identify the efficient allocation which maximizes aggregate advertiser surplus under complete information, and suggest VCG implementation since that allocation generally is not supported by GSP. In GSP, there may exist multiple monotone equilibria where the equilibrium bid functions, now with two arguments m and v, are monotone in both m and v. A future analysis of this kind of model would be both interesting and challenging. Although the heterogeneity of profits per match may render the existence of the monotone equilibria more difficult in some cases, the main message of our model will not be altered, i.e., externalities generally contribute to the existence.

Quality scores

A search engine can also be more active even under the restriction of the particular auction format, e.g., by assigning different weights to different advertisers. A popular practice is to use so-called quality score adjustments for bids and payments. For example, Google uses a combination of historical CTRs, relevance and landing-page user experience of advertisers to weight bids and adjust payments. Denote the quality score by q_k and the payment per click by p_k for an advertiser listed on position k. The advertisers are ranked by the product of their bid and quality score assigned ($\beta \times q$). The price per click for the bidder on position k becomes the lowest bid that still ensures that he will win that position, i.e., $p_k q_k = \beta_{k+1} q_{k+1}$ or $p_k = \frac{\beta_{k+1} q_{k+1}}{q_k}$. It is more interesting to study the effects of quality score adjustments in the model with advertisers of multidimensional types, e.g., heterogeneous profits per click and heterogeneous match rates, and consumers with search costs, since it is then natural for search engines to weight bids in order to trade off consumer participation and revenue.[16] On one hand, the search engine has an incentive to prioritize bidders with high profits per match (and possibly low match rates for positions other than the last one) to extract more rents from bidders. On the other hand, in order to induce more consumer participation, search engines also have an incentive to assign more weight to bidders with high match rates.

Because search engines typically have the ability to estimate the match rates of bidders quite precisely, and these estimates can be used to construct quality scores, to some extent the quality scores can be generally perceived as positively correlated with match rates.[17] To gain insight into the effects of quality scores in our model, consider a scenario where the quality scores are perfectly aligned with the match

[16] Hagiu and Jullien (2011) study a similar trade-off in a model with predetermined prices per click for each bidder.

[17] At the early stage of sponsored link auctions, the historical CTR was used as the main weight factor. Athey and Ellison (2011) discuss potential social efficiency loss with such weighting, due to different clicking probabilities of different bidders with different degrees of speciality. As competition between search engines has become more intense, "relevance" becomes a more important component in order to ensure consumer participation. On the other hand, under the rule of weighting by historical CTRs bidders with a higher match rate also have more incentive to bid more aggressively in order to suffer fewer externalities from others and obtain a higher historical CTR.

rates , i.e., $q = m$. Intuitively, weighting by type discriminates against the low-type bidders. Thus it should force the lower-type bidders to bid more aggressively. Meanwhile in efficient equilibrium, every bidder who wins a position receives a payment discount, which also induces more aggressive bidding and even overbidding possibly. Recalling that non-monotonicity in the full support case is due to the fact that low-type bidders bid too low, we might expect that such quality score adjustments tend to make the existence more robust. For distributions with partial support on the low types, the problem of non-existence may or may not be more severe, depending on which of the discrimination effects or the discount effects dominate.

Revenue equivalence

With endogenous CTRs, an advertiser's payoff from a position below the first position depends on the match rates of other advertisers listed on upper positions. It seems that revenue equivalence does not hold for other standard auctions in such an environment. For example, in the corresponding Generalized English auction with the second price property, the auction starts at an initial bid of zero and it gradually increases. A bidder can drop out at any bids, and if he is the k_{th}-to-last bidder at that bid level, his advertisement is listed on the k_{th} position. Then he pays the bid at which the previous bidder dropped out, for every click he receives. In such an auction, the bidders see the bid history and know the number of remaining bidders and updates his belief about the distribution of the match rates of the remaining bidders. This is quite similar to the situation in a model with interdependent values and revenue equivalence may no longer hold. The following reasoning shows such intuition is false. Even with externalities, all the standard auctions deliver the same allocation as GSP in efficient equilibrium. Hence, in the corresponding direct truthful mechanism, the expected payment from a type m bidder must be tailored such that incentive compatibility conditions take the same form as equation (6) in Appendix A. Therefore, the expected payments in these auctions differ at most by a constant. The revenue equivalence result in this multi-object environment is reminiscent of the same result in the single-object environment in which it holds as long as the the signals are independently distributed.

Optimal Reserve price

The optimal reserve price r^* can be shown to be a value satisfying $r^* = \frac{1-F(r^*)}{f(r^*)}$, which takes the traditional form from the literature.[18] Because the optimal reserve price eliminates the lowest bidders, one would expect that this would contribute to the existence of equilibrium when the distribution of match rates has a full support. However, as our numerical approximation results suggest, this only shifts the bid function upwards, and does not change its shape. Hence the optimal reserve price only plays the traditional role of enhancing revenue.

[18] Gomes and Sweeney (2014) derive the same result as well.

Non-existence of symmetric inefficient equilibrium

Using the same arguments as in Gomes and Sweeney (2014), it can be shown that if there are no symmetric efficient equilibria, then there are no symmetric inefficient equilibria either. Note that our model also satisfies the single-crossing condition which implies there are no mixed strategy equilibria. The pooling equilibria can also be excluded using the same argument, i.e., if a type m bidder is pooling with some types around him, he is actually interested in competing with lower types and he could win the competition simply by a downward ε deviation.

7 Conclusion

We study GSP with allocative externalities that originate from matching between advertisers and consumers in an incomplete information model. Symmetric efficient equilibria are characterized if they exist, and the necessary and sufficient conditions are provided. We give various sufficient conditions on the primitives for the existence of equilibrium when there are 2 positions, which shows that existence is quite general. More importantly, by changing to the more appropriate scope of looking at GSP with externalities, we find that scenarios of non-existence are unlikely and that the intuition for the revenue effect of non-existence is more natural. That is, the source of non-existence and low revenue is the fact that the auction is full of very low-type bidders and thus the externalities virtually disappear.

Appendices

A Proof of Proposition 1

There are several possible approaches that can be used to derive the candidate equilibrium bidding strategies. For example, without appealing to the revelation principle, one can derive the first order condition from maximizing the expected payoff of a bidder with a match rate m but pretending to have a match rate $\hat{m}$, and then use the fact that in efficient equilibrium $\hat{m} = m$ to derive the candidate bid function. The approach here is similar to the one in Gomes and Sweeney (2014)[19], because it is more straightforward.

The efficient equilibrium expected payment in GSP is

$$\begin{aligned} E\big[P^{GSP}(m)\big] &= \sum_{k=1}^{K} \pi_k(m) c_k(m) E\Big[\beta(F^{k+1:N}) | F^{k+1:N} \le m \le F^{k-1:N}\Big] \\ &= (N-1)\int_0^m \beta(t) \sum_{k=1}^{K} c_k(m)(1-F(m))^{k-1}\binom{N-2}{k-1} F(t)^{N-k-1} f(t)\,\mathrm{d}t. \end{aligned} \tag{4}$$

[19]As demonstrated below, appealing to the Integral Form of Envelope theorem is unnecessary.

The derivative can then be derived,

$$\frac{\mathrm{d}}{\mathrm{d}m}E\big[P^{GSP}(m)\big] = (N-1)\Big\{\beta(m)\sum_{k=1}^{K} c_k(m)(1-F(m))^{k-1}\binom{N-1}{k-1}(N-k)F(m)^{N-k-1}f(m)$$
$$-\int_0^m \beta(t)\sum_{k=1}^{K}\Big[c_k(m)(k-1)f(m) - \frac{\mathrm{d}c_k(m)}{\mathrm{d}m}(1-F(m))\Big]$$
$$\times (1-F(m))^{k-2}\binom{N-2}{k-1}F(t)^{N-k-1}f(t)\ \mathrm{d}t\Big\}. \quad (5)$$

By revelation principle, there is a direct truthful mechanism with the same outcome as in the efficient equilibrium of GSP. Denote the expected payment of a bidder with match rate m in the efficient equilibrium of the direct truthful mechanism by $E[P^{DT}(m)]$. Then $E[P^{DT}(m)]$ must satisfy

$$m \in \arg\max_{\hat{m}} \sum_{k=1}^{K} \pi_k(\hat{m})c_k(\hat{m})m - E\big[P^{DT}(\hat{m})\big] \quad (6)$$

which implies that m solves the first order condition from the optimization problem, i.e.,

$$\sum_{k=1}^{K}\Big[\frac{\mathrm{d}\pi_k(m)}{\mathrm{d}m}c_k(m) + \pi_k(m)\frac{\mathrm{d}c_k(m)}{\mathrm{d}m}\Big]m = \frac{\mathrm{d}}{\mathrm{d}m}E\big[P^{DT}(m)\big]. \quad (7)$$

By construction $E[P^{DT}(m)] = E[P^{GSP}(m)]$ for any m, which implies $\frac{\mathrm{d}}{\mathrm{d}m}E\big[P^{DT}(m)\big] = \frac{\mathrm{d}}{\mathrm{d}m}E\big[P^{GSP}(m)\big]$. Equating (5) and (7) and rearranging the resulting equation, we find the equilibrium bidding strategy $\beta(m)$ satisfies a Volterra equation of the second kind

$$\beta(m) = \Phi(m) + \int_0^m Q_1(m,t)\beta(t)\ \mathrm{d}t, \quad (8)$$

where $\Phi(m)$ and $Q_1(m)$ are defined in equation (3). According to Debnath and Mikusiński (1990), equation (8) has a unique solution in (3) provided the technical conditions in equation (3) are satisfied.

The payoff function of bidders satisfies the strict single crossing differences. Hence, if $\beta(m)$ is strictly increasing in m, then by the Constraint Simplification Theorem (Milgrom, 2004, p. 105) we can conclude that a unique efficient equilibrium exists if and only if the bid function in equation (3) is strictly increasing.

B Proof of Proposition 2

We divide the proof of proposition 2 into two steps.

step 1: A critical quantity, $\Lambda(m)$, for the monotonicity of $\beta(m)$.

When $K = 2$, equation (8) becomes

$$\beta(m) = \frac{\phi(m)}{A(m)} + \frac{q(m)}{A(m)}\int_0^m (N-2)F(t)^{N-3}f(t)\beta(t)\ \mathrm{d}t, \quad (9)$$

where

$$A(m) = F(m)^{N-2} + (N-2)F(m)^{N-3}\int_m^1 (1-t)f(t)\,\mathrm{d}t,$$
$$\phi(m) = m^2F(m)^{N-2} + (N-2)mF(m)^{N-3}\int_m^1 (1-t)f(t)\,\mathrm{d}t = m\left[A(m) - (1-m)F(m)^{N-2}\right],$$
$$q(m) = 1-m.$$

Multiplying both sides of equation (9) by $A(m)$, differentiating with respect to m and inserting for $\int_0^m (N-2)F(t)^{N-3}f(t)\beta(t)\,\mathrm{d}t$, we have a first order differential equation for $\beta(m)$,

$$A(m)\beta'(m) = \left[\phi'(m) - \phi(m)\frac{q'(m)}{q(m)}\right] + \left[(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)}\right]\beta(m). \tag{10}$$

It is straightforward to verify that $\beta'(0) = 1$ and $\beta'(1) = 2$. Hence, for the strict monotonicity of $\beta(m)$ we only need to be concerned with $m \in (0,1)$. It is also clear that $A(m) > 0, \phi(m) \geq 0$ and $q(m) \geq 0$ for any m. To study the monotonicity conditions, we divide types into two groups: $(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)}$ is strictly positive and non-positive.

For those m such that $(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)} \leq 0$, $\beta'(m) > 0$ if

$$V(m) \equiv \left[\phi'(m) - \phi(m)\frac{q'(m)}{q(m)}\right] + \left[(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A\frac{q'(m)}{q(m)}\right]m > 0.$$

since in GSP overbidding is weakly dominated by truth-telling.[20] Obviously, $q'(m) = -1$. Because

$$A'(m) = (N-2)mF(m)^{N-3}f(m) + (N-2)(N-3)F(m)^{N-4}f(m)\int_m^1 (1-t)f(t)\,\mathrm{d}t,$$
$$\phi'(m) = \left[A(m) - (1-m)F(m)^{N-2}\right] + m\left[A'(m) + F(m)^{N-2} - (1-m)(N-2)F(m)^{N-3}f(m)\right],$$

we have

$$\begin{aligned}
V(m) &= \left[A(m) - (1-m)F(m)^{N-2}\right] + m\left[A'(m) + F(m)^{N-2} - (1-m)(N-2)F(m)^{N-3}f(m)\right] \\
&\quad - \phi(m)\frac{q'(m)}{q(m)} + m\left[(1-m)(N-2)F(m)^{N-3}f(m) - A'(m) + A(m)\frac{q'(m)}{q(m)}\right] \\
&= A(m) - (1-m)F(m)^{N-2} + mF(m)^{N-2} - \phi(m)\frac{q'(m)}{q(m)} + mA(m)\frac{q'(m)}{q(m)} \\
&= 2mF(m)^{N-2} + (N-2)F(m)^{N-3}\int_m^1 (1-t)f(t)\,\mathrm{d}t + \frac{\phi(m) - mA(m)}{q(m)} \\
&= 2mF(m)^{N-2} + (N-2)F(m)^{N-3}\int_m^1 (1-t)f(t)\,\mathrm{d}t + \frac{mA(m) - m(1-m)F(m)^{N-2} - mA(m)}{1-m} \\
&= mF(m)^{N-2} + (N-2)F(m)^{N-3}\int_m^1 (1-t)f(t)\,\mathrm{d}t.
\end{aligned}$$

Clearly, $V(m) > 0$ and thus $\beta'(m) > 0$ for any bidders with match rates m in this group.

[20] The argument is the same as in single-object second price auctions.

For those m such that $(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)} > 0$, $\beta'(m) > 0$ if $\phi'(m) \geq 0$,[21] since $q'(m) = -1$ and $\phi(m) > 0$ for $m \in (0,1)$. It is found that

$$\begin{aligned}\frac{1}{F(m)^{N-3}}\phi'(m) &= 2mF(m) + (N-2)\int_m^1 (1-t)f(t)\,\mathrm{d}t - m(N-2)(1-2m)f(m) \\ &\quad + m(N-2)(N-3)F(m)^{-1}f(m)\int_m^1 (1-t)f(t)\,\mathrm{d}t \\ &\geq (N-2)\left[\int_m^1 (1-t)f(t)\,\mathrm{d}t - m(1-2m)f(m)\right].\end{aligned} \tag{11}$$

Obviously, if $m \geq \frac{1}{2}$, then $\phi'(m) \geq 0$. Thus, $\beta'(m) > 0$ if, for $m \in [0, \frac{1}{2}]$,

$$\Lambda(m) \equiv \int_m^1 (1-t)f(t)\,\mathrm{d}t - m(1-2m)f(m) \geq 0.$$

step 2: $\Lambda(m) \geq 0$ **for familiar distributions**

First we show the sufficiency of the first two conditions for the existence of the equilibrium. If $f(m)$ is inverse U shaped, or increasing, or (single-peaked) left-skewed, then $f(t) \geq f(m)$ for $t \in [m, 1-m]$. Because

$$\begin{aligned}\int_m^1 (1-t)f(t)\,\mathrm{d}t &= \int_m^{1-m} (1-t)f(t)\,\mathrm{d}t + \int_{1-m}^1 (1-t)f(t)\,\mathrm{d}t \\ &\geq f(m)\int_m^{1-m}(1-t)\,\mathrm{d}t \\ &= f(m)\left[-\tfrac{1}{2}(1-t)^2|_m^{1-m}\right] \\ &= \tfrac{1}{2}f(m)\left[(1-m)^2 - m^2\right] \\ &= \tfrac{1}{2}(1-2m)f(m),\end{aligned}$$

we have

$$\begin{aligned}\Lambda(m) &\geq \tfrac{1}{2}(1-2m)f(m) - m(1-2m)f(m) \\ &\geq 0,\end{aligned}$$

for $m \in [0, \frac{1}{2}]$. Thus, $\beta'(m) > 0$.

Next, consider the case that the density function of the match rates is decreasing and belongs to the power family, i.e., $F(m) = m^a, 0 < a < 1$. Let

$$\lambda(m) \equiv (1+a)\Lambda(m) = a(2a+3)m^{a+1} - (a+1)^2 m^a + 1.$$

[21] One can use a sharper condition here to obtain more general existence results, e.g., $\left[\phi'(m) - \phi(m)\frac{q'(m)}{q(m)}\right] + \left[(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A\frac{q'(m)}{q(m)}\right]\frac{\phi(m)}{A(m)} > 0$ since equation (9) implies $\beta(m) \geq \frac{\phi(m)}{A(m)}$. However, it will only change the results quantitatively, e.g., higher ratio of $\frac{b}{a}$ in the right skewed Beta family, and it complicates the analysis a lot.

Then $\Lambda(m) \geq 0$ if $\lambda(m) \geq 0$. For any $a \in (0,1)$, $\lambda(m)$ is convex.[22] It has a unique stationary point

$$\bar{m}(a) = \frac{a+1}{2a+3} < 1, \quad \bar{\lambda}(a) \equiv \lambda(\bar{m}(a)) = 1 - (a+1)\left(\frac{a+1}{2a+3}\right)^a.$$

It is routine to verify that $\bar{\lambda}'(a) = \left(\frac{a+1}{2a+3}\right)^{a+1}\left[(2a+3)\log\left(\frac{2a+3}{a+1}\right) - 3\right] > 0$ for $a > 0$.[23] And $\bar{\lambda}(0) = 0$ implies that $\bar{\lambda}(a) > 0$ for any $a > 0$. Thus, in this case, $\beta'(m) > 0$.

Finally, consider the standard Beta distribution with parameters $a > 0$ and $b > 0$, i.e.,

$$f(m) = \frac{m^{a-1}(1-m)^{b-1}}{B(a,b)}, \quad B_m(a,b) = \int_0^m x^{a-1}(1-x)^{b-1}dx, \quad B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx.$$

Then

$$\Lambda(m) = \frac{(1-m)B_{1-m}(a,b+1) - (1-m)B_m(a,b+1) - (1-2m)m^a(1-m)^b}{(1-m)B(a,b)},$$

and

$$\Lambda'(m) = \frac{-m^a(1-m)^b\left[m^2(2a+2b+1) - m(3a+b+3) + a + 1\right] - (1-m)^{a+1}m^{b+1}}{m(1-m)^2B(a,b)}.$$

If $a \geq b > 0$, then for $m \in [0, \frac{1}{2}]$,

$$\frac{(1-m)^{a+1}m^{b+1}}{(1-m)^{b+1}m^{a+1}} = \left(\frac{1-m}{m}\right)^{\frac{a}{b}} \geq 1 \Longrightarrow (1-m)^{a+1}m^{b+1} \geq (1-m)^{b+1}m^{a+1}.$$

Thus,

$$\begin{aligned}
\Lambda'(m) &\leq \frac{-m^a(1-m)^b\left(m^2(2a+2b+1) - m(3a+b+3) + a + 1\right) - (1-m)^{b+1}m^{a+1}}{m(1-m)^2B(a,b)} \\
&= -\frac{m^a(1-m)^b}{m(1-m)^2B(a,b)}\left[m^2(2a+2b+1) - m(3a+b+3) + a + 1 + m(1-m)\right] \\
&= -\frac{m^a(1-m)^b}{m(1-m)^2B(a,b)}\Big\{(1-m)\left[1 + a(1-2m)\right] - m\left[1 + b(1-2m)\right]\Big\} \\
&\leq 0.
\end{aligned}$$

Since $\Lambda\left(\frac{1}{2}\right) = 0$ implies $\Lambda(m) \geq 0$, we have $\beta'(m) > 0$.

C Proof of Proposition 3

Because

$$\begin{aligned}
\frac{1}{F(m)^{N-3}}\phi'(m) &= 2mF(m) + (N-2)\int_m^1 (1-t)f(t)\,\mathrm{d}t - m(N-2)(1-2m)f(m) \\
&\quad + m(N-2)(N-3)F(m)^{-1}f(m)\int_m^1 (1-t)f(t)\,\mathrm{d}t \\
&\geq m(N-2)F(m)^{-1}f(m)\int_m^1 (1-t)f(t)\,\mathrm{d}t\left[(N-3) - \frac{(1-2m)F(m)}{\int_m^1 (1-t)f(t)\,\mathrm{d}t}\right],
\end{aligned}$$

[22] The second derivative of $\lambda(m)$ is $a(a+1)m^{a-2}\left[a^2(2m-1) + 3am + 1\right]$. The derivative of $\delta(m) \equiv a^2(2m-1) + 3am + 1$ is $3a + 2a^2 > 0$. Thus the minimum of $\delta(m)$ is achieved when $m = 0$ and $\delta(0) = 1 - a^2 > 0$ for $0 < a < 1$.

[23] The derivative of $\rho(a) \equiv (2a+3)\log\left(\frac{2a+3}{a+1}\right) - 3$ is $2\log\left(\frac{2a+3}{a+1}\right) - \frac{1}{a+1}$ and $\rho''(a) = \frac{1}{(a+1)^2(2a+3)} > 0$. Because $\rho'(0) = 2\log(3) - 1 > 0$ implies $\rho'(a) > 0$ for $a > 0$ and $\rho(0) = 3\log(3) - 3 > 0$, we have $\rho(a) > 0$ and $\bar{\lambda}'(a) > 0$ for $a > 0$.

$\phi'(m) \geq 0$ if $(N-3) - \frac{(1-2m)F(m)}{\int_m^1 (1-t)f(t)\ \mathrm{d}t} \geq 0$. For all $m \in [0, \frac{1}{2}]$, $\frac{(1-2m)F(m)}{\int_m^1 (1-t)f(t)\ \mathrm{d}t}$ is finite if there is no such $\tilde{m} \in [0, \frac{1}{2}]$ that for $m > \tilde{m}$, $f(m) = 0$. If there is such a $\tilde{m}$, then that means that the support of $f(m)$ is reduced to $[0, \tilde{m}]$ and only at $\tilde{m}$, $\frac{(1-2m)F(m)}{\int_m^1 (1-t)f(t)\ \mathrm{d}t}$ is not well defined. But then, at $\tilde{m}$, $(N-2)F(m)^{N-3}f(m)q(m)$ becomes $(N-2)F(\tilde{m})^{N-3}f(\tilde{m})(\tilde{m}-\tilde{m}) = 0$ at $\tilde{m}$ accordingly, and $(N-2)F(m)^{N-3}f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q} < 0$. Thus, if $N - 3 \geq \left\lceil \max_{m \in [0, \frac{1}{2}]} \frac{(1-2m)F(m)}{\int_m^1 (1-t)f(t)\ \mathrm{d}t} \right\rceil$, where $\lceil \cdot \rceil$ is the ceiling function, monotonicity always holds.

D $\beta'(m) \leq 0$ for some m if $F(m)$ is uniform on $[0, \frac{1}{10}]$.

Suppose $F(m)$ is uniform on $[0, b]$. When $K = 2, N = 3$, equation (8) becomes

$$\beta(m) = \frac{\phi(m)}{A(m)} + \frac{q(m)}{A(m)} \int_0^m f(t)\beta(t)\ \mathrm{d}t,$$

where

$$\begin{aligned}
A(m) &= F(m) + \int_m^b (1-t)f(t)\ \mathrm{d}t, \\
\phi(m) &= m^2 F(m) + m \int_m^b (1-t)f(t)\ \mathrm{d}t = m\left[A(m) - (1-m)F(m)\right], \\
q(m) &= 1 - m.
\end{aligned}$$

Observe that $\beta(m) \leq \frac{\phi(m)}{A(m)} + \frac{q(m)}{A(m)} \int_0^m f(t)t\ \mathrm{d}t$, since overbidding is dominated by truth-telling. The Volterra equation above implies

$$A(m)\beta'(m) = \left[\phi'(m) - \phi(m)\frac{q'(m)}{q(m)}\right] + \left[f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)}\right]\beta(m).$$

If $F(m)$ is uniform on $[0, b]$, then

$$f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)} = \frac{m^2 - 4m + 2 + b^2 - 2b}{2b(1-m)}.$$

Function $S(m) \equiv m^2 - 4m + 2 + b^2 - 2b$ has a unique stationary point at $m = 2$ such that $S'(m) < 0$ for $m \leq b \leq 1$ and the minimum of $S(m)$ is achieved at $m = b$ and $S(b) = 2(b^2 - 3b + 1)$. $S(b)$ has two roots $b = \frac{1}{2}\left(3 \pm \sqrt{5}\right)$. For $b \leq \frac{1}{2}\left(3 - \sqrt{5}\right) \approx 0.38$, $S(b) \geq 0$ which implies $S(m) \geq 0$ for any $m \in [0, \frac{1}{10}]$ when $b = \frac{1}{10}$. Then $\beta'(m) \leq 0$ if

$$V(m) \equiv \left[\phi'(m) - \phi(m)\frac{q'(m)}{q(m)}\right] + \left[f(m)q(m) - A'(m) + A(m)\frac{q'(m)}{q(m)}\right]\left[\frac{\phi(m)}{A(m)} + \frac{q(m)}{A(m)}\int_0^m f(t)t\ \mathrm{d}t\right] \leq 0.$$

It can be verified that

$$V(m) = \frac{30000m^7 - 80000m^6 + 47600m^5 - 59000m^4 + 58461m^3 - 7422m^2 - 4161m + 361}{20(1-m)\left(100m^2 + 19\right)}.$$

Because $V(0) > 0$ and $V(\frac{1}{10}) < 0$, $V(m) < 0$ for some $m \in [0, \frac{1}{10}]$. Hence, $\beta(m)$ is not monotone and the efficient equilibrium fails to exist.

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