Optimal Prize Allocations in Contests with Maximal performance objective

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January 14, 2018

Abstract

We show that weak concavity of the cost function leads to optimality of single prize in contests with maximal performance objective, which generalizes the previous result in Chawla et al. (2015). Moldovanu and Sela (2001) show that, with the constant elasticity functional form, enough convexity can provide a rationale for multiple prizes under total performance objective. Surprisingly, we find optimality of single prize continues to hold for arbitrary degree of convexity under maximal performance objective when the number of contestants is three. On the contrary, if the cost function is piecewise linear, then the convexity argument for multi-prize can be restored. Furthermore, We find that in terms of optimal prize allocations there is an interesting relationship between the two objectives. In the derivation of the results, a series of simple facts about the winning probability functions is presented, which may be useful for future works in contest theory and multi-object auction theory.

Keywords: Contests; Optimal architecture; Prizes; Incentives

JEL codes: D44; C78; L86; J31

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We thank Espen R. Moen, Jingfeng Lu, Dana Sisak, Etienne Wasmer, Steffen Grønneberg, Eivind Eriksen, Plamen Nenov, seminar participants at Norwegian Business School and participants of the 2017 CBESS Conference on Contests: Theory and Evidence for helpful comments. The usual disclaimer applies.

1 Introduction

Instead of contracts, contests may be used for procurements (either private or public), or more generally incentivizing agents to produce better performances, with the advantage of mitigating moral hazard problems (but may induce collusion). The contest designer, or the principal, typically has the freedom of choosing a multi-prize allocation or simply a single prize allocation, which immediately leads to the following question: should there be multiple prizes or just one prize, or when to set a single prize and when multiple prizes? Clearly the answer depends on the objective of the principal.

In many situations, only the best performance is useful for the principal and hence the objective is to maximize the *maximal performance*. Research tournaments, sponsored either by governments or private companies, are typically meant to find the most efficient solutions for practical problems, e.g., vaccines, engines, standards of technological products, and so on. As a concrete example, in 2006 Netflix announced a million dollar challenge for a recommendation algorithm (to recognize customers' preferences and recommend contents accordingly) that can improve the current one by 10%. The emergence of internet even makes it possible for individuals to post questions on Q&A web sites and reward the best answer (with coveted virtual coins) among the ones created by large numbers of talented internet users.

The pioneering work by Chawla et al. (2015) shows that if the cost of producing performance is linear, then it is optimal for the principal to set a single prize. We show that the single-prize optimality can be extended to weak concave cost functions, using some simple facts about the win probability functions together with an amplification lemma.

The answer for the convex case is more subtle. Under another common objective, the *total performance* objective where the principal enjoys the total performances of all contestants, Moldovanu and Sela (2001) show that a convex-enough cost function (e.g., based on the familiar Arrow-Pratt coefficient) can provide a rationale for multi-prize allocations. At the first glance, the two kinds of objectives appear to be very different in terms of optimal prize allocation. Intuitively, if the cost function is convex enough, on the margin it can be too costly to incentivize the champion to produce performance and "cheaper to buy" marginal units of performance from the runners-up. Hence, the optimality of multiple prizes under total performance objective follows naturally since the principal is indifferent among whom the marginal performance is produced from. On the other hand, under maximal performance objective, only the champion's performance is relevant and the performances of the runners-up are simply wasted, one may expect that it should always be optimal to keep the whole prize budget for the champion. Indeed, with the same convex cost function (constant elastic) as in Moldovanu and Sela (2001), we show that single-prize optimality continues to hold independent of the degree of convexity when the number of the contestants is three. On the contrary, we also find that if the cost function is piecewise linear, then optimality of multi-prize allocations can be restored if it is convex enough. Hence, not only the convexity but also the exact functional form is relevant for determining the optimal number of prizes.

The results above appear to suggest that there is a connection between the two objectives in terms of optimal prize allocations. Indeed, we find: if it is optimal to set a single prize under total performance objective, then it is also optimal to set a single prize for the maximal performance objective; The contrapositive statement is also true, i.e., if it is optimal to set multiple prizes for the maximal performance objective, then it is also optimal to set multiple prizes for the total performance objective.

In addition, we show that the main results above can be smoothly derived by using a series of simple facts about the winning probability functions, which may be useful for similar analyses in contest theory and multi-object auction theory.

The rest of the paper is organized as follows. A brief review of related literature is given for the rest of this section. In section 2 We describe the model and characterize the equilibrium. In section 3 we then identify the optimality of single prize for weak concave cost functions. In section 4, by considering constant elastic and piecewise linear cost functions respectively, we show the subtlety of the role of convexity. In section 5 the relationship between the two objectives (maximal and total performance) is identified. Finally, section 6 concludes.

Related literature

The earlier works on contests with maximal performance objective typically take a certain prize allocation as given. Given a single prize, Nalebuff and Stiglitz (1983), Taylor (1995) and Fullerton and McAfee (1999) find that restricting contestant entry is beneficial for the principal under various settings. Still assuming a single prize (size being endogenously chosen by contestants), Che and Gale (2003) find the similar results for maximal surplus objective (performance net of prize). Considering both maximal and total performance objective, Moldovanu and Sela (2006) study a two-stage contest and compare the outcome from the two-round contest to the outcome from a single-round contest with a single prize. Assuming an exogenous minimal performance threshold, Megidish and Sela (2013) compare the outcome from a contest with a single prize to the outcome in a stylized random contest. They find if the threshold is too high then the random contest may induce more total performance and maximal performance.

Instead of fixed prize allocations, Kaplan et al. (2002) and Kaplan et al. (2003) consider particular forms of performance-dependent reward allocations and find qualitatively different behavior of contestants under incomplete and complete information. Cohen et al. (2008) study optimal performance-dependent reward allocations and they find that the optimal reward may be decreasing in performance and there is no possibility for optimality of multiple rewards.

There has been a large literature in contest theory studying contests with noised performances following the pioneering work on rent-seeking by Tullock (1980). There is also a number of studies of all-pay auctions with complete information, e.g.,Baye et al. (1996), Barut and Kovenock (1998), Clark and Riis (1998) and Glazer and Hassin (1988). These works are however more remote to ours since the performances are not noised in our incomplete information setting.

A more detailed and excellent survey for the literature on optimal prize allocation in general is provided by Sisak (2009).

2 The model

A risk-neutral principal initiates a contest in which there are N risk-neutral contestants. The contestants differ in endowed proficiency of a certain skill relevant for the contest, which are independent draws from a common distribution $F(s)$ with a density function $f(s) > 0$ on $[\underline{s}, \bar{s}]$. For convenience, we denote the inverse function of $F(s)$ simply by $s(F)$ and similarly the inverse function of $F(x)$ by $x(F)$. To produce performance q, a contestant with skill s incurs a non-recoverable cost which takes a separable functional form $\frac{1}{A(s)}B(q)$ with $A(\cdot) > 0, B'(\cdot) > 0$. It is natural that given a certain q, a more skilled contestant is more cost efficient, i.e., $A'(\cdot) > 0$.

A total budget of \$1 for the contest is divided into $N-1$ prizes, denoted $a_1, \cdots, a_{N-1}$, i.e., $\sum_{n=1}^{N-1} a_n = 1$.[1] Without loss of generality, we assume $a_1 \geq a_2 \geq \cdots \geq a_{N-1} \geq 0$. The rule of the contest is such that the contestant with the best performance wins the first prize a_1 and the contestant with the second highest performance wins the second prize a_2, and so on.

A unique symmetric monotone equilibrium can be derived by the following standard procedure. In the equilibrium the probability of a contestant with skill s winning the n_{th} prize, a_n, is

$$\pi_n(s) = \binom{N-1}{n-1}(1-F(s))^{n-1}F(s)^{N-n} \equiv \mathcal{P}_n(F(s)), \tag{1}$$

which implies

$$\frac{d}{ds}\pi_n(s) \quad = \quad \frac{d\mathcal{P}_n(F(s))}{dF(s)}f(s).$$

It is clear that $\mathcal{P}_n(0) = 0$ for $n = 1, \cdots, N-1$. Denote the equilibrium performance function by $q(s)$. The incentive compatibility condition,

$$s = \arg\max_{\hat{s}} \quad \sum_{n=1}^{N-1} \pi_n(\hat{s})a_n - \frac{1}{A(s)}B(q(\hat{s}))$$

implies that the equilibrium performance function must satisfy

$$\sum_{n=1}^{N-1} \pi_n'(s)a_n = \frac{1}{A(s)}B'(q(s))q'(s). \tag{2}$$

Since $B'(\cdot) > 0$, the monotonicity of $q(s)$ is guaranteed if $\sum_{n=1}^{N-1} \pi_n'(s)a_n \geq 0$, which is shown to be true below.

The following definition is repeatedly used in the rest of this paper.

Definition 1 (Single-crossing)**.** A function defined on a certain interval has single crossing property if it crosses zero only once and from below. If function g_1 crosses another function g_2 only once and from below, then we say g_1 single-crosses g_2.

Lemma 1. $\sum_{n=1}^{N-1} \pi_n'(s)a_n \geq 0$ *or* $\sum_{n=1}^{N-1} \mathcal{P}_n'(F)a_n \geq 0$.

[1] It is obvious that a positive amount of prize for the lowest rank is a waste of money since anyone ranked the lowest will earn that prize and it does not provide any incentive.

Sketch of Proof. The proof uses the simple fact that $\sum_{n=1}^{N} \mathcal{P}_n(F) = 1$ for any F, which simply says that any contestant can win one prize for sure if there are N prizes available. This implies $\sum_{n=1}^{N} \mathcal{P}'_n(F) = 0$ (which in turn implies the fact $\sum_{n=1}^{N} \mathcal{P}''_n(F) = 0$ which is used in the proof of Lemma 6). Combined with the discrete single-crossing property of sequence $\{\mathcal{P}'_n(F)\}$ (for any given F), a discrete version of the amplification lemma in the next section can be used to establish the statement because $\{a_n\}$ is a non-increasing sequence. See Appendix A for details. □

Because $A'(s) > 0$, the single crossing difference condition (from Athey (2001) or Milgrom (2004)) for the sufficiency of equilibrium existence is satisfied. With the initial condition $q(0) = 0$, the solution to the differential equation in (2) is unique and can be derived straightforwardly from (2).

Theorem 1. *There exists a unique symmetric increasing equilibrium in the contest. The equilibrium performance function is*

$$q(s) = B^{-1}\Big(\int_{\underline{s}}^{s} A(x) \sum_{n=1}^{N-1} \pi'_n(x) a_n dx\Big). \tag{3}$$

3 Optimality of single prize for weak concave cost functions

Given a prize allocation $\{a_n\}$, the expected performance from the champion, i.e., the principal's expected revenue under maximal performance objective, is $E[q(F^{1:N})] = \int_{\underline{s}}^{\bar{s}} q(s) N F(s)^{N-1} f(s) ds$, where $F^{1:N}$ represents the first highest order statistics of N independent draws from distribution F. On the other hand, the expected total performance from all contestants is simply $N \int_{\underline{s}}^{\bar{s}} q(s) f(s) ds$. Hence the expected revenue for the two objectives can be written as, for $i \in \{0, N-1\}$,

$$\begin{aligned} R_i &= N \int_{\underline{s}}^{\bar{s}} q(s) F(s)^i f(s) ds \\ &= N \int_{\underline{s}}^{\bar{s}} B^{-1}\Big(\int_{\underline{s}}^{s} A(x) \sum_{n=1}^{N-1} \pi'_n(x) a_n dx\Big) F(s)^i f(s) ds. \end{aligned} \tag{4}$$

The principal chooses $\{a_n\}$ to maximize R_i under respective objective.

Throughout the proofs of this paper, the following intuitive argument is repeatedly used, which we state as a lemma.

Lemma 2 (Amplification lemma). *On an interval (a, b), suppose $g(F)$ has the single-crossing property. Then for a non-decreasing function $h(F) \geq 0$, we have*

$$\int_a^b g(F) dF \geq 0 \implies \int_a^b g(F) h(F) dF \geq 0.$$

If we define $g(F) = g_1(F) - g_2(F)$, then it is clear that the following is also true: if a function $g_1(F)$ single-crosses another function $g_2(F)$, then

$$\int_a^b (g_1(F) - g_2(F)) dF \geq 0 \implies \int_a^b (g_1(F) - g_2(F)) h(F) dF \geq 0.$$

The following lemma gives important insights for the optimality of single prize (for both the two common objectives).

Lemma 3. *On the open unit interval* $(0,1)$*, for* $j \in \{2, \cdots, N-1\}$*, as a function of* F*,*

- $\mathcal{P}_1(F) - \mathcal{P}_j(F)$ *has the single-crossing property, or* $\mathcal{P}_1(F)$ *single-crosses* $\mathcal{P}_j(F)$*.*
- $\mathcal{P}_1'(F) - \mathcal{P}_j'(F)$ *has the single-crossing property, or* $\mathcal{P}_1'(F)$ *single-crosses* $\mathcal{P}_j'(F)$*.*
- *Let* F_j^* *be the point at which* $\mathcal{P}_1'(F) - \mathcal{P}_j'(F) = 0$ *and* F_j^{**} *the point at which* $\mathcal{P}_1(F) - \mathcal{P}_j(F) = 0$*. Then* $F_j^{**} \geq F_j^*$*.*

Proof. See Appendix B. □

Figure 1 illustrates Lemma 3 nicely when $N = 5$.

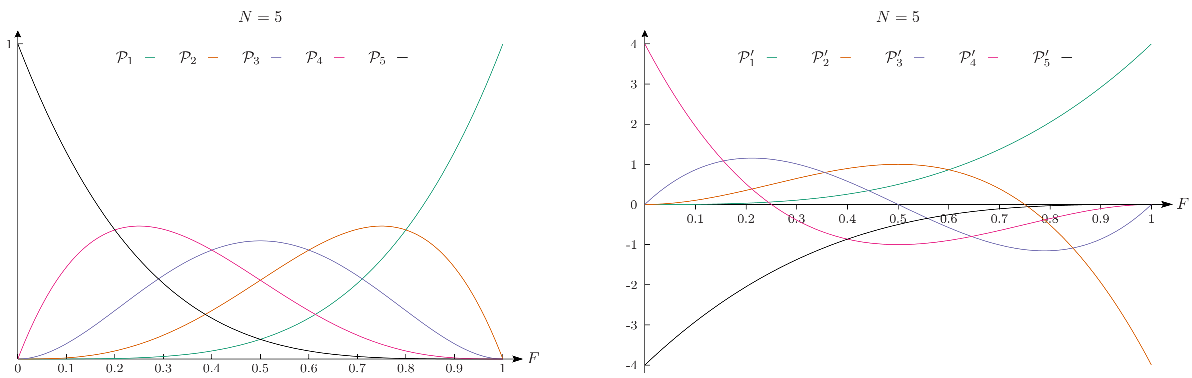

Figure 1: $\mathcal{P}_n(F)$ and $\mathcal{P}_n'(F)$ when $N = 5$, $n = 1, \cdots, N$.

In addition, there is a simple fact about the *ex-ante* probability of winning nth prize for a contestant: for any n,

$$\int_0^1 \mathcal{P}_n(F) dF = \frac{1}{N}. \tag{5}$$

Immediately it follows that

$$\int_0^1 \left(\mathcal{P}_1(F) - \mathcal{P}_j(F)\right) dF = \int_0^1 \int_0^F \left(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\right) d\mathcal{F} dF = 0, \tag{6}$$

since $\mathcal{P}_n(0) = 0$ for any $n \neq N$. Combining the amplification lemma, Lemma 3 and (6), we have the following lemma.

Lemma 4. *For a non-negative function* $\mathcal{A}(F)$ *with* $\mathcal{A}'(F) > 0$ *on the open unit interval,* $\int_0^F \mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big) d\mathcal{F}$*, as a function of* F*, has the single-crossing property. Moreover,*

$$\int_0^1 \int_0^F \mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big) d\mathcal{F} dF \geq 0. \tag{7}$$

Sketch of Proof. Let F_j^* be the one defined in Lemma (3). The amplification lemma can be applied with the help of a new function defined as follows,

$$\bar{\mathcal{A}}_j(F) = \begin{cases} \mathcal{A}(F) & \text{if } F \leq F_j^* \\ \mathcal{A}(F_j^*) & \text{if } F > F_j^* \end{cases},$$

with which the reader may already be able to visualize the geometry of a complete proof. See Appendix C for details. □

By the amplification lemma again and Lemma 4, the following proposition is easy to show.

Proposition 1. *If $B(q)$ is concave or linear, then it is optimal to set a single prize for both total performance objective and maximal performance objective.*

Proof. Following (4), for $i \in \{0, N-1\}$, let

$$\tau_j(\varepsilon) \equiv \int_{\underline{s}}^{\bar{s}} B^{-1}\left(\int_{\underline{s}}^{s} A(x)\left[\pi_1'(x)(a_1+\varepsilon) + \pi_j'(x)(a_j-\varepsilon) + \sum_{n\neq 1,j}^{N-1} \pi_n'(x)a_n\right]dx\right) F(s)^i f(s)ds. \quad (8)$$

Then

$$\left.\frac{d}{d\varepsilon}\tau_j(\varepsilon)\right|_{\varepsilon=0} = \int_{\underline{s}}^{\bar{s}} \frac{1}{B'(q(s))} \int_{\underline{s}}^{s} A(x)\big(\pi_1'(x) - \pi_j'(x)\big)dxF(s)^i f(s)ds.$$

The inverse function of distribution function $F(s)$ is denoted by $s(F)$ and $F(x)$ by $x(F)$. By change of variable method,

$$\left.\frac{d}{d\varepsilon}\tau_j(\varepsilon)\right|_{\varepsilon=0} = \int_0^1 \frac{1}{B'(q(s(F)))} \int_0^F A(x(\mathcal{F}))\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}F^i dF. \quad (9)$$

Because $A'(x) > 0$, $A(x(\mathcal{F}))$ is increasing in $\mathcal{F}$. By Lemma 4, $\int_0^F A(x(\mathcal{F}))\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}F^i$ also crosses zero only once and from below. If $B(q)$ is concave or linear, then $\frac{d}{dF}B'(q(s(F))) = \frac{d}{dq}B'(q(s(F)))\frac{d}{ds}q(s)\frac{d}{dF}s(F) \leq 0$. That is, the inverse of $B'(q(s(F)))$ is non-decreasing in F and non-negative. Moreover, F^i is also non-decreasing and non-negative. By the amplification lemma and Lemma 4, we have $\frac{d}{d\varepsilon}\tau_j(\varepsilon)\big|_{\varepsilon=0} \geq 0$ and this completes the proof. □

Proposition 1 shows that optimality of single prize under maximal performance objective is valid for any separable cost function with weak concavity, and hence extends the result in Chawla et al. (2015) in which $B(q) = q$ in their model.

4 Convex cost functions

Moldovanu and Sela (2001) find that it may be optimal to set multiple prizes under total performance objective if the cost function is convex enough in effort. It is interesting to know wether it may also be optimal to use a multi-prize allocation to maximize the maximal performance.

4.1 Incentivizing the champion by rewarding the champion only

In this direction, the role of convexity is more subtle, as we show below that if the cost function is constant elastic (CE) in performance then there is no role for convexity when the number of the contestants is three.

Lemma 5. *For any* $\{a_n\}$ *where* $a_1 \neq 1$, $\frac{\sum_{n=1}^{N-1} \mathcal{P}_n'(F)a_n}{\sum_{n=1}^{N-1} \mathcal{P}_n(F)a_n} < \frac{\mathcal{P}_1'(F)}{\mathcal{P}_1(F)}$ *and* $\frac{d}{dF}\left(\frac{\mathcal{P}_1(F)}{\sum_{n=1}^{N-1} \mathcal{P}_n(F)a_n}\right) > 0$.

Proof. See Appendix D. □

Proposition 2. *Consider the case where* $C(s,q) = \frac{1}{A(s)} q^{\frac{1}{\sigma}}, \sigma > 0$. *If there are only 3 contestants, it is always optimal to set a single prize for a principal with maximal performance objective.*

Proof. See Appendix E. □

The result above may sound surprising, given the fact that the cost function can exhibit arbitrary degree of convexity in performance. In particular, Moldovanu and Sela (2001) have given an example of optimality of multi-prize allocations under total performance objective with $\sigma = \frac{1}{2}$, our result shows the fact that the qualitative result of optimal prize allocation problem under maximal performance objective can be very different from the one under total performance objective.

4.2 Incentivizing the champion by rewarding the runners-up

From Lemma 4, $\int_0^F A(x(\mathcal{F}))\left[\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_2'(\mathcal{F})\right] d\mathcal{F}$ has single crossing property on the open unit interval. For a given $A(\cdot)$ function, let $\bar{F}_2$ be the point at which $\int_0^{\bar{F}_2} A(x(\mathcal{F}))\left[\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\right] d\mathcal{F} = 0$.

We now show that if the cost function is piecewise linear in performance, then it can be optimal to have multiple prizes for the principal with either objective. Specifically, consider the following case[2]

$$B(q) = \begin{cases} q & 0 \leq q \leq \bar{q}_2 \\ \theta q - \lambda & q > \bar{q}_2 \end{cases}, \tag{10}$$

where $\lambda = (\theta - 1)\bar{q}_2$, and $\bar{q}_2 = q(s(\bar{F}_2))$.[3] By the convexity assumption $\theta > 0$ and $\lambda > 0$.

Following (4), define

$$\tau_2(\varepsilon) \equiv \int_{\underline{s}}^{\bar{s}} B^{-1}\left(\int_{\underline{s}}^{s} A(x)\left[\pi_1'(x)(1-\varepsilon) + \pi_2'(x)(0+\varepsilon) + \sum_{n\neq 1,2}^{N-1} \pi_n'(x)a_n\right]dx\right) F(s)^{N-1} f(s) ds.$$

[2]The assumed piecewise linear function $B(\cdot)$ has a kink at q_2^*, i.e., not differentiable at q_2^*. However, one can always smooth the kink away by defining a smooth function on a small segment containing q_2^* such that $B(\cdot)$ is differentiable everywhere and the derivative is continuous. And the segment can be arbitrarily small such that it does not affect the qualitative conclusion below. For the purpose of clear exposition, we take the limiting case as legal.

[3]Since we have a degree of freedom from λ, for any q_2^*, there is always a corresponding λ such that $B(\cdot)$ is continuous at $\bar{q}_2$.

Then

$$
\begin{aligned}
-\frac{d}{d\varepsilon}\tau_2(\varepsilon)\Big|_{\varepsilon=0} &= \int_0^1 \frac{1}{B'(q(s(F)))}\int_0^F A(x(\mathcal{F}))\Big[\mathcal{P}_1'(\mathcal{F})-\mathcal{P}_2'(\mathcal{F})\Big]d\mathcal{F}F^{N-1}dF \\
&= \int_0^{\bar{F}_2}\int_0^F A(x(\mathcal{F}))\Big[\mathcal{P}_1'(\mathcal{F})-\mathcal{P}_2'(\mathcal{F})\Big]d\mathcal{F}F^{N-1}dF \\
&\quad +\frac{1}{\theta}\int_{\bar{F}_2}^1\int_0^F A(x(\mathcal{F}))\Big[\mathcal{P}_1'(\mathcal{F})-\mathcal{P}_2'(\mathcal{F})\Big]d\mathcal{F}F^{N-1}dF.
\end{aligned}
$$

By the definition of $\bar{F}_2$ above, the first double integral is negative while the second one is positive. Since both integrals are finite, there exists a $\bar{\theta}$ such that for $\theta \geq \bar{\theta}$, $-\frac{d}{d\varepsilon}\tau_2(\varepsilon) < 0$. We then have the following result.

Proposition 3. *If the $B(\cdot)$ function takes the form as in (10), then a multi-prize allocation is optimal for both objectives when θ is large enough.*

The results from the CE case in the previous section and this piecewise linear case reveal the fact that not only the convexity but also its exact form is important for the optimality prize allocations with maximal performance objective.

5 The relationship between the two objectives

An interesting observation from Proposition 1 and Proposition 3 is that the optimality condition of single prize or multi-prize is valid for both objectives under the similar qualitative condition. In Proposition 2 we find that in the CE case convexity is irrelevant for the optimal prize allocation for maximal performance objective when there are 3 contestants. On the other hand, enough convexity can lead to optimality of multi-prize for total performance objective even in that case. One may wonder if there is any relationship between the two objectives in terms of optimal prize allocations. Indeed, we show below that the following interesting relationship holds.

Lemma 6. *For $F \in (0,1)$, $\frac{d}{dF}\frac{\mathcal{P}_1'(F)}{\sum_{n=1}^{N-1}\mathcal{P}_n'(F)a_n} > 0$ and hence $\mathcal{P}_1'(F)$ single-crosses $\sum_{n=1}^{N-1}\mathcal{P}_n'(F)a_n$.*

Proof. See Appendix F. □

Proposition 4. *If it is optimal to set a single prize when the objective is total performance, then it is also optimal to set a single prize for maximal performance objective. On the other hand, if it is optimal to set multiple positive prizes when the objective is maximal performance objective, then it is also optimal to set multiple positive prizes for total performance objective.*

Proof. The first statement in the proposition says that if

$$\int_0^1 B^{-1}\Big(\int_0^F A(x(\mathcal{F}))\Big[\mathcal{P}_1'(\mathcal{F})\Big]d\mathcal{F}\Big)dF \geq \int_0^1 B^{-1}\Big(\int_0^F A(x(\mathcal{F}))\Big[\sum_{n=1}^{N-1}\mathcal{P}_n'(\mathcal{F})a_n\Big]d\mathcal{F}\Big)dF$$

then

$$\int_0^1 B^{-1}\Big(\int_0^F A(x(\mathcal{F}))\Big[\mathcal{P}_1'(\mathcal{F})\Big]d\mathcal{F}\Big)F^{N-1}dF \geq \int_0^1 B^{-1}\Big(\int_0^F A(x(\mathcal{F}))\Big[\sum_{n=1}^{N-1}\mathcal{P}_n'(\mathcal{F})a_n\Big]d\mathcal{F}\Big)F^{N-1}dF.$$

Obviously, by the amplification lemma the statement above is true if $B^{-1}\big(\int_0^F A(x(\mathcal{F}))\big[\mathcal{P}_1'(\mathcal{F})\big]d\mathcal{F}\big)$ single-crosses $B^{-1}\big(\int_0^F A(x(\mathcal{F}))\big[\sum_{n=1}^{N-1}\mathcal{P}_n'(\mathcal{F})a_n\big]d\mathcal{F}\big)$. The single-crossing condition can be met if $\int_0^F A(x(\mathcal{F}))\big[\mathcal{P}_1'(\mathcal{F})\big]d\mathcal{F}$ single-crosses $\int_0^F A(x(\mathcal{F}))\big[\sum_{n=1}^{N-1}\mathcal{P}_n'(\mathcal{F})a_n\big]d\mathcal{F}$. Since Lemma 6 shows that $\mathcal{P}_1'(F)$ single-crosses $\sum_{n=1}^{N-1}\mathcal{P}_n'(F)a_n$, by the spirit of the geometry from Lemma 3 to Lemma 4, the single crossing condition is satisfied and so the first statement is true. The second statement is contrapositive to the first one and this completes the proof.

□

6 Conclusion

By studying a standard model in a private information environment, we generalize the previous result for optimality of different prize allocations under maximal performance objective. We also find that optimality of multi-prize is also possible even under maximal performance objective. A comparison of different forms of cost function illustrates the importance of the elasticity of costs for the optimal prize allocation. In terms of optimal prize allocation, an interesting relationship between the two objectives is found. Findings on the properties of the win probability functions may be useful for future works.

Appendices

A Proof of Lemma 1

First,

$$\sum_{n=1}^{N-1} \pi_n'(s) a_n = f(s) \sum_{n=1}^{N-1} \frac{d\mathcal{P}_n(F(s))}{dF(s)} \geq 0, \text{ if } \sum_{n=1}^{N-1} \frac{d\mathcal{P}_n(F(s))}{dF(s)} \geq 0.$$

It is clear to see that

$$\mathcal{P}_n(F) = \binom{N-1}{n-1}(1-F)^{n-1}F^{N-n}, \sum_{n=1}^{N} \mathcal{P}_n(F) = 1, \sum_{n=1}^{N} \mathcal{P}_n'(F) = 0, \sum_{n=1}^{N-1} \mathcal{P}_n'(F) \geq 0$$

since $\mathcal{P}_N'(F) \leq 0$ for any F. Also, for $n \in \{1, \cdots, N-1\}$,

$$\mathcal{P}_n'(F) = \binom{N-1}{n-1}(1-F)^{n-2}F^{N-n-1}\big(N-n-(N-1)F\big).$$

For any F, if and only if $n > (N-1)(1-F)+1$, $\mathcal{P}_n'(F) < 0$. For a given F, let $\mathcal{P}_{n^*}'(F)$ be the first negative element in the sequence $\{\mathcal{P}_n'(F)\}_{n=1}^{N}$. For $a_1 \geq \cdots \geq a_N \geq 0$, $\sum_{n=1}^{N-1} \mathcal{P}_n'(F)a_n = \sum_{n=1}^{n^*-1} \mathcal{P}_n'(F)a_n + \sum_{n=n^*}^{N-1} \mathcal{P}_n'(F)a_n \geq \sum_{n=1}^{n^*-1} \mathcal{P}_n'(F)a_{n^*} + \sum_{n=n^*}^{N-1} \mathcal{P}_n'(F)a_{n^*} = a_{n^*} \sum_{n=1}^{N-1} \mathcal{P}_n'(F) \geq 0$. The argument above is an application of the discrete version of the amplification lemma.

B Proof of Lemma 3

First we show that $\mathcal{P}_1(F) - \mathcal{P}_j(F)$ crosses zero only once and from below, i.e., $\mathcal{P}_1(F)$ and $\mathcal{P}_j(F)$ intercept with each other only once on the open unit interval.

$$\begin{aligned} \mathcal{P}_1(F) - \mathcal{P}_j(F) &= F^{N-1} - \binom{N-1}{j-1}(1-F)^{j-1}F^{N-j} \\ &= F^{N-j}\left[F^{j-1} - \binom{N-1}{j-1}(1-F)^{j-1}\right] \\ &\equiv F^{N-j}\big(F^{j-1} - [\theta(1-F)]^{j-1}\big). \end{aligned}$$

where $\theta = \binom{N-1}{j-1}^{1/(j-1)} \geq 1$. Obviously, F^{j-1} intercepts $[\theta(1-F)]^{j-1}$ only once at $F^* = \frac{\theta}{1+\theta} < 1$ and from below.

Second, it is straightforward to derive

$$\begin{aligned} \mathcal{P}_j'(F) &= \binom{N-1}{j-1}(1-F)^{j-2}F^{N-j-1}\big((N-j)-(N-1)F\big), \\ \mathcal{P}_1'(F) - \mathcal{P}_j'(F) &= (N-1)F^{N-j-1}\left[F^{j-1} - \binom{N-1}{j-1}(1-F)^{j-2}\Big(\frac{N-j}{N-1} - F\Big)\right]. \end{aligned}$$

Define

$$G(F) \equiv \binom{N-1}{j-1}(1-F)^{j-2}\left(\frac{N-j}{N-1} - F\right)$$

Then

$$G'(F) = -\binom{N-1}{j-1}(1-F)^{j-3}\left[\left(1-\frac{N-j}{N-1}\right)+(j-1)\left(\frac{N-j}{N-1}-F\right)\right]$$

Function $G(F) \lesseqgtr 0$ for $F \gtreqless \frac{N-j}{N-1}$ and $G'(F) \leq 0$ for $F \leq \frac{N-j}{N-1}$. Hence, $G(F)$ is positive and decreasing on $[0, \frac{N-j}{N-1}]$ and negative on $[\frac{N-j}{N-1}, 1]$, while F^{j-1} is always increasing and has a value zero at the origin. Thus, there is a unique point F^* at which $\frac{d}{dF}\big(\mathcal{P}_1(F) - \mathcal{P}_j(F)\big) = 0$. And for $F \gtreqless F^*$, $\frac{d}{dF}\big(\mathcal{P}_1(F) - \mathcal{P}_j(F)\big) \gtreqless 0$.

Finally, when $\frac{d}{dF}\big(\mathcal{P}_1(F_j^*) - \mathcal{P}_j(F_j^*)\big) = 0$, $\mathcal{P}_1(F_j^*) - \mathcal{P}_j(F_j^*) \leq 0$. By the first results above, we have $F_j^{**} \geq F_j^*$.

C Proof of Lemma 4

From (6),

$$\int_0^1 \big(\mathcal{P}_1(F) - \mathcal{P}_j(F)\big)dF = \int_0^1 \int_0^F \big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}dF = 0.$$

By assumption $\mathcal{A}'(F) > 0$. From Lemma 3 we know that both $\mathcal{P}_1(F) - \mathcal{P}_j(F)$ and $\mathcal{P}_1'(F) - \mathcal{P}_j'(F)$ have the single-crossing property. Let F_j^* be the point at which $\mathcal{P}_1'(F) - \mathcal{P}_j'(F) = 0$. Define

$$\bar{\mathcal{A}}_j(F) \equiv \begin{cases} \mathcal{A}(F) & \text{if } F \leq F_j^* \\ \mathcal{A}(F_j^*) & \text{if } F > F_j^*. \end{cases}$$

Then by the amplification lemma,

$$\int_0^1 \bar{\mathcal{A}}_j(F)\big(\mathcal{P}_1(F) - \mathcal{P}_j(F)\big)dF \geq 0, \tag{11}$$

because $\bar{\mathcal{A}}_j(s)$ is a non-decreasing function. Inequality (12) is equivalent to

$$\int_0^1 \left[\int_0^F \bar{\mathcal{A}}_j(F)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}\right]dF \geq 0,$$

since $\mathcal{P}_n(0) = 0$ for $n = 1, \cdots, N-1$.

Recall the single-crossing property of $\mathcal{P}_1'(F) - \mathcal{P}_j'(F)$ and the definition of F_j^*.

When $\mathcal{F} \leq F \leq F_j^*$, $\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F}) \leq 0$ and $\mathcal{A}(\mathcal{F}) \leq \mathcal{A}(F)$, which implies $\mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big) \geq \bar{\mathcal{A}}_j(F)\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)$. Hence, for any $F \leq F_j^*$,

$$\int_0^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} \geq \int_0^F \bar{\mathcal{A}}_j(F)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}. \tag{12}$$

For any $F > F_j^*$, $\bar{\mathcal{A}}_j(F) = \mathcal{A}(F_j^*)$ and we write

$$\begin{aligned} \int_0^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} &= \int_0^{F_j^*} \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} + \int_{F_j^*}^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}, \\ \int_0^F \bar{\mathcal{A}}_j(F)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} &= \int_0^{F_j^*} \mathcal{A}(F_j^*)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} + \int_{F_j^*}^F \mathcal{A}(F_j^*)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}. \end{aligned}$$

When $\mathcal{F} > F_j^*$, $\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F}) > 0$ and $\mathcal{A}(\mathcal{F}) > \mathcal{A}(F)$, which implies $\mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big) > \bar{\mathcal{A}}_j(F_j^*)\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)$. Hence, for any $F > F_j^*$ and consequently for any F, inequality (13) holds. Therefore, for any F and any j,

$$\int_0^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} \;\geq\; \int_0^F \bar{\mathcal{A}}_j(F)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F},$$

which implies

$$\begin{aligned}\int_0^1\int_0^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}dF \;&\geq\; \int_0^1\int_0^F \bar{\mathcal{A}}_j(F)\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}dF \\ &\geq\; 0.\end{aligned}$$

That is, (7) is true.

Furthermore, it is obvious that $\int_0^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F}$ first crosses zero at some point when F increases from 0 to 1, say at $\tilde{F}_j$. We must have that for $F < \tilde{F}_j$ the integral is negative and $\mathcal{P}_1'(\tilde{\mathcal{F}}_j) - \mathcal{P}_j'(\tilde{\mathcal{F}}_j) > 0$ by the single-crossing property of $\mathcal{P}_1'(F) - \mathcal{P}_j'(F)$ and the non-negativity of $\mathcal{A}(F)$.[4] Thus for $F > \tilde{F}_j$, $\int_0^F \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} > 0$ by the single-crossing property of $\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})$, which completes the proof.

D Proof of Lemma 5

Because $\frac{\mathcal{P}_1'(F)}{\mathcal{P}_1(F)} = \frac{N-1}{F}$, the first inequality is equivalent to

$$F\sum_{n=1}^{N-1}\mathcal{P}_n'(F)a_n \;<\; (N-1)\sum_{n=1}^{N-1}\mathcal{P}_n(F)a_n.$$

For each $n \geq 2$,

$$\frac{F\mathcal{P}_n'(F)a_n - (N-1)\mathcal{P}_n(F)a_n}{\binom{N-1}{n-1}(1-F)^{n-2}F^{N-n}a_n} \;=\; 1-n < 0.$$

Therefore, the first inequality holds and the second one follows immediately since both $\mathcal{P}_1(F)$ and $\sum_{n=1}^{N-1}\mathcal{P}_n(F)a_n$ are positive.

E Proof of Proposition 2

When $B(q) = q^{\frac{1}{\sigma}}$, the equilibrium $q(s)$ can be solved as

$$q(s) = \left(\int_0^s A(x)\sum_{n=1}^{N-1}\pi_n'(x)a_n dx\right)^{\sigma} \tag{13}$$

Let $\mathcal{A}(F) \equiv A(s(F))$, which is increasing in F. The expected performance of the champion is

$$\int_0^1\left[\int_0^F \mathcal{A}(\mathcal{F})\left(\sum_{n=1}^{N-1}\mathcal{P}_n'(\mathcal{F})a_n\right)d\mathcal{F}\right]^{\sigma}F^{N-1}dF. \tag{14}$$

[4] If $\mathcal{P}_1'(\tilde{\mathcal{F}}) - \mathcal{P}_j'(\tilde{\mathcal{F}}) \leq 0$, then $\int_0^{\tilde{F}} \mathcal{A}(\mathcal{F})\Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\Big)d\mathcal{F} < 0$, which contradicts the definition of $\tilde{F}$.

Let

$$\tau_j(\varepsilon) = \int_0^1 \left[\int_0^F \mathcal{A}(\mathcal{F}) \Big(\mathcal{P}_1'(\mathcal{F})(a_1+\varepsilon) + \mathcal{P}_j'(\mathcal{F})(a_j-\varepsilon) + \sum_{n\neq 1,j}^{N} \mathcal{P}_n'(\mathcal{F}) a_n \Big) d\mathcal{F} \right]^\sigma F^{N-1} dF$$

Then the inequality is true if for every $j \in \{2, \cdots, N-1\}$,

$$\left.\frac{d\tau_j(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^1 \sigma \left(\int_0^F \mathcal{A}(\mathcal{F}) \big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F}) a_n \big) d\mathcal{F} \right)^{\sigma-1} \int_0^F \mathcal{A}(\mathcal{F}) \Big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F}) \Big) d\mathcal{F} F^{N-1} dF \geq 0.$$

Lemma 4 implies that $\left(\int_0^F \mathcal{A}(\mathcal{F})\big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F})a_n\big)d\mathcal{F}\right)^{-1} \int_0^F \mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}F^{N-1}$, as a function of F, also has the single-crossing property. Because $\left(\int_0^F \mathcal{A}(\mathcal{F})\big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F})a_n\big)d\mathcal{F}\right)^\sigma$ is positive and nondecreasing in F, by the amplification lemma, the inequality is true in this case if

$$\int_0^1 \frac{\int_0^F \mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}}{\int_0^F \mathcal{A}(\mathcal{F})\big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F})a_n\big)d\mathcal{F}} F^{N-1} dF \geq 0. \tag{15}$$

Recall that $\mathcal{P}_1(F) = F^{N-1}$. Therefore, by Lemma 3, equation (6), Lemma 5 and the amplification lemma,

$$\int_0^1 \frac{\int_0^F \big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}}{\int_0^F \big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F})a_n\big)d\mathcal{F}} F^{N-1} dF = \int_0^1 \left(\mathcal{P}_1(F) - \mathcal{P}_j(F)\right) \frac{\mathcal{P}_1(F)}{\sum_{n=1}^{N-1} \mathcal{P}_n(F) a_n} dF \geq 0. \tag{16}$$

Thus (16) is true if for every F, if

$$\frac{\int_0^F \mathcal{A}(\mathcal{F})\big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}}{\int_0^F \mathcal{A}(\mathcal{F})\big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F})a_n\big)d\mathcal{F}} \geq \frac{\int_0^F \big(\mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_j'(\mathcal{F})\big)d\mathcal{F}}{\int_0^F \big(\sum_{n=1}^{N-1} \mathcal{P}_n'(\mathcal{F})a_n\big)d\mathcal{F}}. \tag{17}$$

Consider the situation where the principal only consider two prizes at most although there are N contestants, i.e., $a_1 + a_2 = 1$. Notice that for contests with only 3 contestants the optimal prize allocation will never offer more than two positive prizes. In order to prove inequality (18) for this situation, we introduce a useful theorem from Wijsman (1985).

Theorem 2 (Wijsman's inequality)**.** *Let μ be a measure on the real line $\mathcal{R}$ and let f_i, g_i $(i = 1, 2)$ be four Borel-measurable functions: $\mathcal{R} \to \mathcal{R}$ such that $f_2 \geq 0, g_2 \geq 0$, and $\int f_i g_j d\mu < \infty$ $(i, j = 1, 2)$. If f_1/f_2 and g_1/g_2 are monotonic in the same direction, then*

$$\int f_1 g_1 d\mu \int f_2 g_2 d\mu \geq \int f_1 g_2 d\mu \int f_2 g_1 d\mu.$$

If in addition, $\int f_1 g_2 d\mu > 0$ and $\int f_2 g_2 d\mu > 0$, then

$$\frac{\int f_1 g_1 d\mu}{\int f_1 g_2 d\mu} \geq \frac{\int f_2 g_1 d\mu}{\int f_2 g_2 d\mu}.$$

In our case, let

$$f_1 = \mathcal{A}(\mathcal{F}), g_1 = \mathcal{P}_1'(\mathcal{F}) - \mathcal{P}_2'(\mathcal{F}), f_2 = 1, g_2 = \mathcal{P}_1'(\mathcal{F}) a_1 + \mathcal{P}_2'(\mathcal{F})(1 - a_1).$$

Obviously, $\int f_1 g_2 d\mu > 0$ and $\int f_2 g_2 d\mu > 0$ sine we have already shown in Lemma 1 that $\sum_{n=1}^{N-1} \mathcal{P}'_n(F)a_n > 0$ for any F. Then because $f_1/f_2 = \mathcal{A}(\mathcal{F})$ is non-decreasing, (18) is true if

$$\frac{d}{dF}\left(\frac{\mathcal{P}'_1(F) - \mathcal{P}'_2(F)}{\mathcal{P}'_1(F)a_1 + \mathcal{P}'_2(F)(1-a_1)}\right) \geq 0.$$

It can be verified that

$$\frac{d}{dF}\left(\frac{\mathcal{P}'_1(F) - \mathcal{P}'_2(F)}{\mathcal{P}'_1(F)a_1 + \mathcal{P}'_2(F)(1-a_1)}\right) = \frac{N-2}{[(1-F)(N(1-a_1)-1) + (2a_1-1)]^2} > 0$$

Therefore, when $N = 3$, inequality (15) holds if $\sigma \geq 0$.

F Proof of Lemma 6

Since $\mathcal{P}'_1(F)$ and $\mathcal{P}''_1(F)$ are both nonnegative and $\frac{\mathcal{P}'_1(F)}{\mathcal{P}''_1(F)} = \frac{F}{N-2}$, $\frac{d}{dF}\frac{\mathcal{P}'_1(F)}{\sum_{n=1}^{N-1}\mathcal{P}'_n(F)a_n} > 0$ is equivalent to $\mathcal{P}''_1(F)\sum_{n=1}^{N-1}\mathcal{P}'_n(F)a_n - \mathcal{P}'_1(F)\sum_{n=1}^{N-1}\mathcal{P}''_n(F)a_n > 0$ or

$$\sum_{n=1}^{N-1}[(N-2)\mathcal{P}'_n(F) - F\mathcal{P}''_n(F)]a_n > 0. \tag{18}$$

For each n,

$$(N-2)\mathcal{P}'_n(F) - F\mathcal{P}''_n(F) \quad = \quad \binom{N-1}{n-1}F^{N-n-1}(1-F)^{n-3}(n-1)(N-n-(N-2)F)$$

Hence, for any given F, $(N-2)\mathcal{P}'_n(F) - F\mathcal{P}''_n(F)$ is negative if and only if $n > N-(N-2)F$. And since $\sum_{n=1}^{N}\mathcal{P}'_n(F) = 0$ for any given F, we have $\sum_{n=1}^{N}\mathcal{P}''_n(F) = 0$. Thus,

$$\begin{aligned}\sum_{n=1}^{N-1}\left((N-2)\mathcal{P}'_n(F) - F\mathcal{P}''_n(F)\right) &= -(N-2)\mathcal{P}'_N(F) + F\mathcal{P}''_N(F) \\ &= (N-2)(N-1)(1-F)^{N-2} + F(N-2)(N-1)(1-F)^{N-3} \\ &= (N-2)(N-1)(1-F)^{N-3} \\ &> 0.\end{aligned}$$

Hence, by the same spirit of the discrete version of the amplification lemma as in the proof of Lemma 1, (19) is true and $\frac{d}{dF}\frac{\mathcal{P}'_1(F)}{\sum_{n=1}^{N-1}\mathcal{P}'_n(F)a_n} > 0$.

Observe that $\mathcal{P}'_1(0) = 0 \leq \sum_{n=1}^{N-1}\mathcal{P}'_n(0)a_n = (N-1)a_{N-1}$, and $\mathcal{P}'_1(1) = N-1 > \sum_{n=1}^{N-1}\mathcal{P}'_n(1)a_n = (N-1)(a_1-a_2)$. Hence $\mathcal{P}'_1(F)$ single-crosses $\sum_{n=1}^{N-1}\mathcal{P}'_n(F)a_n$ and this completes the proof.

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